Classical Mechanics
PhD Qualifying Exam, 2018 Spring

1. (30 points) Let $\vec{q} \equiv (q_1, q_2, ..., q_N)$ be the generalized coordinate and $\vec{p} \equiv (p_1, p_2, ..., p_N)$ be the canonical momentum conjugate to $\vec{q}$, respectively. In the Hamiltonian formulation of classical mechanics, one considers all the possible small variations $\delta\vec{q} \equiv \vec{q}(t) - \vec{q}^*(t)$ and $\delta\vec{p} \equiv \vec{p}(t) - \vec{p}^*(t)$ about the true trajectory $(\vec{q}^*(t), \vec{p}^*(t))$ in the phase space, and assumes that

$$S \equiv \int_{t=0}^{t=T} \vec{p} \cdot d\vec{q} - H(\vec{q}, \vec{p}, t)\, dt \equiv \int_{t=0}^{t=T} \left(\sum_{j=1}^{N} p_j dq_j \right) - H(\vec{q}, \vec{p}, t)\, dt$$

is an extremum when the trial trajectory $(\vec{q}(t), \vec{p}(t))$ is set equal to $(\vec{q}^*(t), \vec{p}^*(t))$. In the above, we have assumed that the trial trajectory also satisfies the boundary conditions $\vec{q}(0) = \vec{q}^*(0)$ and $\vec{q}(T) = \vec{q}^*(T)$. This is one version of the "Hamilton's principle."

   (a) (10 points) Please show that Hamilton's principle implies that

$$\begin{aligned} \frac{d\vec{q}^*}{dt} &= \frac{\partial H(\vec{q}^*, \vec{p}^*, t)}{\partial \vec{p}^*}, \\ \frac{d\vec{p}^*}{dt} &= -\frac{\partial H(\vec{q}^*, \vec{p}^*, t)}{\partial \vec{q}^*}. \end{aligned}$$

   (b) (5 points) Please show that H is a constant of the motion if it is known that $\partial H/\partial t = 0$.

   (c) (10 points) Let ε be a small parameter and $\vec{a}$ be some given constant vector, respectively. Assume that H satisfies the following symmetry

$$H(\vec{q}, \vec{p}, t) = H(\vec{q} + \varepsilon\vec{a}, \vec{p}, t) + O(\varepsilon^2),$$

   where $O(\varepsilon^2)$ means something which is of second order in the small parameter ε. Please show that

$$\vec{p}^* \cdot \vec{a} = \text{ a constant of the motion.}$$

   (d) (5 points) We now specialize to the motion of a particle in a three-dimensional space using cylindrical coordinate $\vec{q} \equiv (r, \phi, z)$ and its associated canonical momentum $\vec{p} \equiv (p_r, p_\phi, p_z)$. The particle is moving under the gravitational influence of an infinitely extended helical structure (Fig. 1 on Page 2) whose symmetry axis coincides with the z-axis. The "pitch" of the helix is c. Assuming uniform mass density for the structure, please show that $p_\phi \cdot 2\pi + p_z \cdot c$ is a constant of the motion.

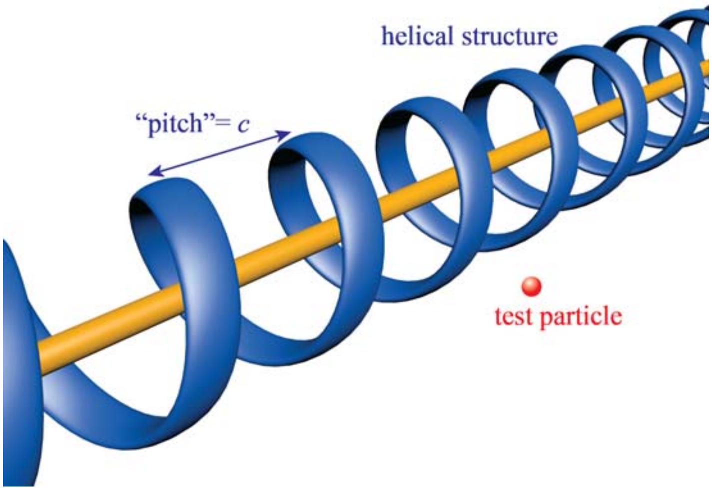

Fig. 1 for Problem 1(d)

2. (40 points) A particle of mass m is constrained to move on the vertical hoop of Fig. 2. The hoop has a radius R, and is spinning at an angular speed ω about its vertical axis. Assume friction can be ignored.

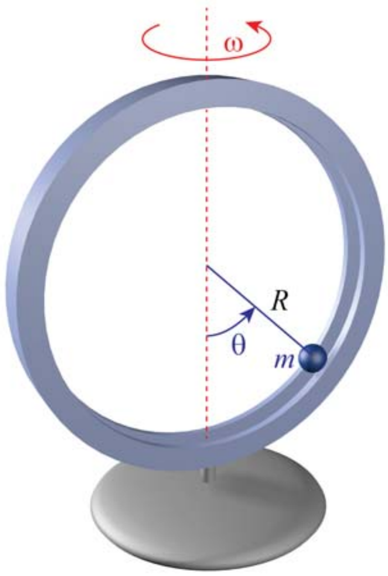

Fig. 2 for Problem 2

   (a) (10 points) Please write down the Lagrangian L of the particle in terms of m, R, ω, θ, $\dot{\theta}$, and the local gravitational acceleration g.
   (b) (10 points) Please write down the Euler-Lagrange equation for the particle.
   (c) (10 points) Please find *all* the possible steady states (that is, solutions whose θ is independent of the time t) of the particle.
   (d) (10 points) Please determine the (linear) stability of the solution(s) of Part 2(c).

3. (30 points) Five identical particles each of mass m are connected by four identical springs of spring constant k to form a linear chain (Fig. 3). We label the particles by the

index -2, -1, 0, 1, and 2, respectively. Let ψ_j denote the small horizontal displacement of the j-th particle from its equilibrium position. Write $\psi \equiv \big(\psi_{-2}, \psi_{-1}, \psi_0, \psi_1, \psi_2\big)$. Define a linear operator $\hat{L}\psi \equiv \big(-\psi_2, -\psi_1, -\psi_0, -\psi_{-1}, -\psi_{-2}\big)$, that is, $\hat{L}$ is a reflection of the displacements of the particles about the center of the linear chain. It is obvious that $\hat{L}\psi$ is a solution to the equations of motion if ψ is. We will make use of this fact to solve this problem.

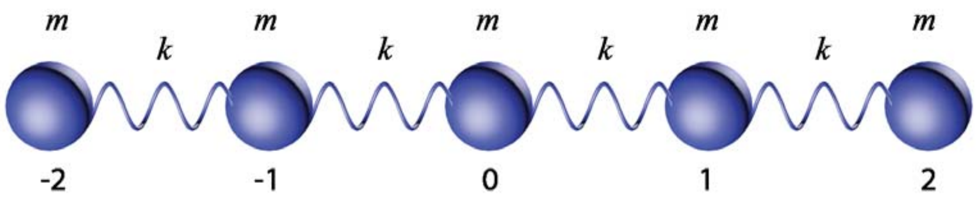

Fig. 3 for Problem 3

(a) (5 points) Show that the eigenvalues of $\hat{L}$ are ± 1. (Hint: Consider $\hat{L}^2$.)

(b) (5 points) Consider the eigenfunction ψ_+ of $\hat{L}$ with the eigenvalue $+1$. Explain why ψ_+ must take the form $\psi_+ = (-y, -x, 0, x, y)$ for some numbers x and y.

(c) (5 points) Write down the equations of motion for Particles 1 and 2 associated with ψ_+. (They are coupled differential equations involving $\ddot{x}$, $\ddot{y}$, x, and y.)

(d) (5 points) Find the normal modes and the corresponding normal frequencies associated with ψ_+.

(e) (10 points) Find the normal modes and the corresponding normal frequencies associated with ψ_-, the eigenfunction of $\hat{L}$ with the eigenvalue -1.