Spring 2018

Qualifying Exam — Quantum Mechanics I

1. Let $\hat{A}$, $\hat{B}$ and $\hat{C}$ be Hermitian operators on some Hilbert space. Assume that operator $\hat{A}$ has a non-degenerate spectrum and that $\hat{A}\hat{B}$ and $\hat{A}\hat{C}$ are also Hermitian.

   (a) (10%) Prove that $[\hat{B}, \hat{C}] = 0$.

   (b) (10%) If $|\psi_1\rangle$ and $|\psi_2\rangle$ are eigenstates of $\hat{B}$ with distinct eigenvalues, show that $\langle\psi_1|\hat{C}|\psi_2\rangle = 0$.

2. Consider a spin-1/2 system quantized along the $+z$ direction, so that the spin operators

$$\hat{S}_i = \frac{\hbar}{2}\sigma_i \quad (i = x, y, z) \, ,$$

   where σ_i are the Pauli matrices. As usual, the base kets in this standard basis are the eigenvectors

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \end{pmatrix} \, .$$

   (a) (10%) Derive the eigenvalues and normalized eigenvectors of the operator $\hat{O} \equiv \hat{S}_x + \hat{S}_y$. Express your answer in the standard basis. Explicitly verify that the two eigenvectors are orthogonal.

   (b) (5%) For the system in the state corresponding to the larger eigenvalue, what is the probability that a measurement of $\hat{S}_z$ yields $+\hbar/2$?

   (c) (10%) Find the unitarity transformation matrix going from the standard basis to the new basis of operator $\hat{O}$ derived above. Verify its unitarity.

   (d) (5%) Derive $\hat{O}$ in the new basis.

3. The one-dimensional simple harmonic oscillator has the Hamiltonian

$$\mathcal{H} = \frac{\mathcal{P}^2}{2m} + \frac{1}{2}m\omega^2\mathcal{X}^2 = \left(a^\dagger a + \frac{1}{2}\right)\hbar\omega$$

   where

$$a \equiv \frac{1}{\sqrt{2}}\left(\frac{\mathcal{X}}{x_0} + i\frac{x_0\mathcal{P}}{\hbar}\right) \, , \quad N \equiv a^\dagger a \, , \text{ and } \quad x_0 = \sqrt{\frac{\hbar}{m\omega}} \, .$$

   Denote the energy eigenstates by $\{|n\rangle\}$ with $N\,|n\rangle = n\,|n\rangle$.

(a) (10%) Start from the canonical commutation relation $[\mathcal{X}, \mathcal{P}] = i\hbar$, derive the following commutation relations:

$$[a, a^\dagger]\ ,\ [a, N]\ ,\ \text{and}\ [a, \mathcal{H}]. \tag{1}$$

(b) (10%) Work out the matrix representations of the following operators: a, $\mathcal{X}$, and $\mathcal{P}$. You are required to show explicitly the matrix elements of the upper-left 4×4 sub-matrix.

(c) (10%) What kinds of transitions between different levels can be induced by the $\mathcal{X}$ and $\mathcal{X}^2$ operators? What are the corresponding energies involved in the transitions of the precious problem?

4. Two spin-1/2 particles form a composite system that interacts with an external vector field $\mathbf{A}$ in such a way that the Hamiltonian is given by

$$H = \mathbf{A} \cdot (\mathbf{S}_1 - \mathbf{S}_2)\ .$$

(a) (10%) Suppose the total spin angular momentum of the composite system is determined at $t = 0$ to be zero. Calculate the normalized state of the system $|\psi(t)\rangle$ at a later time t, as well as the first time $|\psi(t)\rangle$ becomes orthogonal to $|\psi(0)\rangle$.

(b) (10%) Suppose that a state, denoted by $|\alpha\rangle$, has a $+\hbar$ eigenvalue of $S_n \equiv \hat{\mathbf{n}}\cdot\mathbf{S}$ along the direction $\hat{\mathbf{n}} = (\sin\theta, 0, \cos\theta)$. Express this state in terms of the eigenstates of S_z.