# NTU-Physics Statistical Physics Qualifying Exam (2018)

(*Please note: 5 problems on 2 pages; Answers in both Chinese and English are OK.*)
The following formulas may be useful:

(1) $\sum_{\mathbf{k}} \rightarrow \dfrac{L^d}{(2\pi)^d}\int d^d k;\ \sum_{\mathbf{p}} \rightarrow \dfrac{L^d}{h^d}\int d^d p\ \ (V = L^d \rightarrow \infty),\ d$ is the dimensionality of the box.

(2) $I_v \equiv \int_0^\infty e^{-\alpha y^2} y^v dy = \begin{cases} (1/2)\sqrt{\pi/\alpha} & \text{for } v=0 \\ (1/4)\sqrt{\pi/\alpha^3} & \text{for } v=2 \end{cases}$

(3) Bose-Einstein integrals $g_n(z) = \dfrac{1}{\Gamma(n)}\int_0^\infty \dfrac{x^{n-1}dx}{z^{-1}e^x - 1} = \sum_{k=1}^\infty \dfrac{z^k}{k^n}\ \ (0 \le z \le 1),\ \ \Gamma(n+1) = n\Gamma(n)$
If $z = 1\ (\mu = 0),\ g_n(1) = \zeta(n)\ \ n > 1;$ if $z \rightarrow 1,\ g_n(z) \rightarrow \infty\ \ n \le 1.$

(4) $\cosh(x) = (\dfrac{e^x + e^{-x}}{2}),\ \sinh(x) = (\dfrac{e^x - e^{-x}}{2});\ \cosh(x)' = \sinh(x),\ \sinh(x)' = \cosh(x).$

1. (20 %) An ideal classical gas composed of $N$ particles, each of mass $m$, is enclosed in a vertical cylinder of height $L$ placed in a uniform gravitational field (of acceleration $g$) and is in thermal equilibrium; ultimately, both $N$ and $L \rightarrow \infty$.
(a) (12 %) Evaluate the partition function of the gas and derive expressions for its major thermodynamic properties (at least, $U$, $C_V$).
(b) (8 %) Explain why the specific heat of this system is larger than that of a corresponding system in free space.

2. (20 %)
(a) (10 %) Evaluate the partition function and the major thermodynamic properties ($A$, $S$, $U$) of an ideal gas consisting of $N_1$ molecules of mass $m_1$ and $N_2$ molecules of mass $m_2$, confined to a space of volume $V$ at temperature $T$. Assume that the molecules of a given kind are mutually indistinguishable, while those of one kind are distinguishable from those of the other kind.
(b) (10 %) Compare your results with the ones pertaining to an ideal gas consisting of ($N_1 + N_2$) molecules, *all of one kind*, of mass $m$, such that $m(N_1+N_2) = m_1N_1 + m_2N_2$.

3. (20 %) In the harmonic approximation, the lattice vibration energy of a solid can be approximated as the internal energy of a phonon gas in the solid of $V$ and at temperature $T$ given by $U(T,V) = \sum_i \hbar\omega_i (1/2 + \langle \hat{n}_i \rangle)$ where $\langle \hat{n}_i \rangle$ is the average number of phonons in quantum state $i$. Einstein assumed, for simplicity, that all the frequencies are equal, i.e.,

$\omega_i = \omega_E$ .

(a) (7 %) Derive the general expression for the heat capacity $C_V$.

(b) (6 %) Derive the expression for $C_V$ in the Einstein approximation.

(c) (7 %) Calculate $C_V$ in (b) in the high $T$ [$T \gg \theta_E = (\hbar\omega_E / k)$] and low $T$ ($T \ll \theta_E$) limits.

4. (20 %) Consider an ideal Bose gas in an isotropic *two-dimensional* harmonic trap.

(a) (10 %) Determine the density of states $g(\varepsilon)$, and express the number of particles in the excited states, $N_e$, and the number of particles in the ground state, $N_0$.

(b) (10 %) Can a Bose-Einstein condensate form in this trap? If so, what is the Bose-Einstein condensation temperature $T_c$ and the condensate fraction $N_0/N$ for $T \le T_c$? If not, why?

5. (20 %) The partition function of the one-dimensional Ising model is given by $Q_I(B,T) = \sum_{\sigma_1}\sum_{\sigma_2}\cdots\sum_{\sigma_N} \exp[\beta \sum_{k=1}^{N} (J\sigma_k\sigma_{k+1} + B\sigma_k)]$. Introduce a 2×2 transfer matrix $P$ such that $\langle\sigma|P|\sigma'\rangle = \exp\{\beta[J\sigma\sigma' + B(\sigma+\sigma')/2]\}$, where ($\sigma$, $\sigma'$ = ±1). One can solve the 1-D Ising model with this transfer matrix by assuming the periodic boundary condition $\sigma_{N+1} = \sigma_1$.

(a) (8 %) Calculate the partition function $Q_I(B,T)$.

(b) (6 %) Calculate the Helmholtz free energy $A_I(B,T)$.

(c) (6 %) Calculate the magnetization $M_I(B,T)$ and show that for any nonzero temperature, there is no spontaneous magnetization in the one-dimensional Ising model.