Ph.D. Qualifying Exam: Quantum Mechanics (II)

March 2018

Problem 1. (20 points) If an electron is bound to a proton in a state of orbital angular momentum l and is subjected to a spin-orbital Hamiltonian of $\eta\vec{L}\cdot\vec{S}$, where $\vec{L}$ and $\vec{S}$ are the orbital angular momentum and spin operators, respectively, and η is a constant. **(a) (12 points)** Find the total angular momentum *j*-states (eigenstate) in terms of the product states of $|lm_l\rangle$ and $|sm_s\rangle$, where l and s are the values of the orbital and spin angular momenta of the electron, respectively, and $m_l\hbar$ and $m_s\hbar$ are their corresponding z-component values [i.e., find the Clebsch-Gordon (C-G) coefficients for the states of the total angular momentum $\vec{J}=\vec{L}+\vec{S}$ which can have values of $j=l\pm 1/2$] **(If you don't know how to find the answer for the general case of arbitrary value l, you may try to solve the problem for a special case of $l=1$ to get partial credits)**. **(b) (8 points)** Find the corresponding eigenenergies of the electron.

Problem 2. (20 points) Consider a spin-1/2 particle with gyromagnetic ratio γ in a magnetic field $\boldsymbol{B}=B_1\hat{i}+B_0\hat{k}$. **(a)** Treating B_1 as a perturbation, **(i) (6 points)** calculate the first-order and second-order shift in energy and **(ii) (4 points)** calculate the first-order energy shift in wave function for the ground state. **(b) (10 points)** Solve the energy eigenvalue problem exactly, and compare the exact answers expended to the corresponding orders.

Problem 3. (22 points) (a) (10 points) Show that the lowest order of the average transition rate of a system subject to a periodic perturbation $H_1(t)=V_1 e^{-i\omega t}$ is given by the Fermi's golden rule. **(b) (12 points)** A hydrogen atom is in the ground state at $t=-\infty$. An electric field $\boldsymbol{E}(t)=E_0e^{-t^2/\tau^2}\hat{z}$ is applied in the *z*-direction until $t=\infty$. Find, to the first order, the probability that the atom ends up in any of the $n=2$ states. **First few normalized hydrogen atom eigenfunctions are given by :**

$$\begin{cases}\psi_{100}=\left(\dfrac{1}{\pi a_0^3}\right)^{1/2}e^{-r/a_0}, \\ \psi_{200}=\left(\dfrac{1}{32\pi a_0^3}\right)^{1/2}\left(2-\dfrac{r}{a_0}\right)e^{-r/2a_0},\end{cases}\qquad \begin{cases}\psi_{210}=\left(\dfrac{1}{32\pi a_0^3}\right)^{1/2}\left(\dfrac{r}{a_0}\right)e^{-r/2a_0}\cos\theta, \\ \psi_{2,1,\pm1}=\mp\left(\dfrac{1}{64\pi a_0^3}\right)^{1/2}\left(\dfrac{r}{a_0}\right)e^{-r/2a_0}\sin\theta e^{\pm i\phi}.\end{cases}$$

Useful Integral: $\int_0^\infty x^n e^{-\alpha x}dx=\dfrac{n!}{\alpha^{n+1}}$, for $\alpha>0$.

Problem 4. (18 points) (a) (6 points) describe what the Wigner-Eckart theorem is and what its use is. **(b) (6 points)** Why does the description of the spontaneous emission of an atom require quantization of the electromagnetic field? Why is the induced or stimulated emission (in addition to the population inversion of atoms in a cavity) related to the generation principle of a laser? **(c) (6 points)** What is the Aharonov-Bohm effect? Please draw a graph of a double-slit experiment to illustrate and explain your answer.

Problem 5. (20 points) (a) (8 points) Calculate in the Born approximation the scattering amplitude $f_Y(\theta)$ of an incident particle of mass m and kinetic energy $E = \frac{\hbar^2 k^2}{2m}$ on the Yukawa potential $V_Y(r) = g\frac{e^{-\mu_0 r}}{r}$. **(b) (5 points)** Find the differential cross section of the scattering problem in (a), and then show that one may obtain the differential cross section $\left.\frac{d\sigma}{d\Omega}\right|_{\text{Coul}} = \frac{(Ze^2)^2}{16E^2 \sin^4(\theta/2)}$ for Coulomb scattering of a particle of charge e on a potential $V(r) = \frac{Ze^2}{r}$ if one sets $g = Ze^2,\ \mu_0 = 0$. **(c) (7 points)** Assume that the interaction Hamiltonian between two identical spin-$\frac{1}{2}$ neutrons (each with mass m) is $V_Y(r) = g\frac{e^{-\mu_0 r}}{r}$, where r is the magnitude of the relative coordinate between the two identical neutrons. Calculate the differential cross section for *unpolarized* neutron-neutron scattering in the center-of-mass frame **[express the answer in terms of the scattering amplitude $f_Y(\theta)$].**