

**Classical Mechanics**  
**PhD Qualifying Exam, 2019 Spring**

1. (35 points) For the linear chain formed of three springs and two mass points of Fig.1, let  $x_j$  denote the displacement of the  $j$ -th particle from its equilibrium position. The equations of motion for the two particles are

$$\begin{aligned} m \frac{d^2 x_1}{dt^2} &= -Kx_1 - C(x_1 - x_2), \\ m \frac{d^2 x_2}{dt^2} &= -Kx_2 - C(x_2 - x_1). \end{aligned}$$

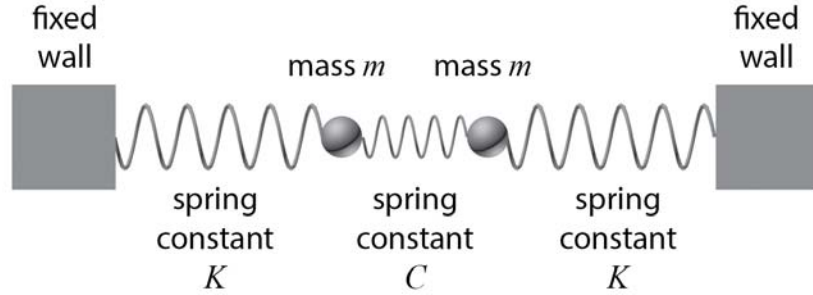


Fig.1: A linear chain formed of springs and mass points.

- (a) (10 points) Please find the normal modes and their associated normal frequencies  $\omega_1$  and  $\omega_2$  ( $\omega_2 > \omega_1$ ) in terms of  $m$ ,  $K$ , and  $C$ .
- (b) (10 points) Suppose initially we have

$$\begin{aligned} x_1(0) &= A, & \dot{x}_1(0) &= 0, \\ x_2(0) &= 0, & \dot{x}_2(0) &= 0 \end{aligned}$$

for some constant  $A$ . Please compute  $x_1(t)$  and  $x_2(t)$  in terms of  $A$ ,  $\omega_1$  and  $\omega_2$ .

- (c) (5 points) Specializing to the case when  $C \ll K$ , then we expect  $\omega_1$  and  $\omega_2$  to be quite close. If we write  $\omega_2 \equiv \omega_1 + \varepsilon$ , please show that the displacement of either particle derived in Part (b) can be cast into the form

$$x_j(t) = (\text{a slowly varying amplitude}) \cdot (\text{a fast oscillating term}).$$

- (d) (5 points) Show that the energy of the system is mostly transferred to particle 2 when the time  $t$  reaches  $\pi/\varepsilon$ ; and the energy is mostly transferred back to particle 1 when  $t$  reaches  $2\pi/\varepsilon$ .
- (e) (5 points) It is easy to envision why particle 1 begins to “feed” its energy to particle 2 during the early time of  $t < \pi/\varepsilon$ , but why does it keep “pumping” its energy to particle 2 during the latter stage of  $t < \pi/\varepsilon$  when its own energy is already smaller than that of particle 2?

2. (40 points) (Lagrangian and Hamiltonian dynamics.) Let  $q \equiv (q_1, q_2, \dots, q_N)$  and  $L(q, \dot{q}, t)$  be the generalized coordinates and the Lagrangian, respectively. Let  $q^*(t)$  denote a true trajectory of a system in the configuration space. If we consider all the trial paths  $q(t)$  satisfying  $q(0) = q^*(0)$  and  $q(T) = q^*(T)$ , then Hamilton's principle dictates that the integral

$$\int_0^T L(q, \dot{q}, t) dt$$

is extremized by the actual trajectory  $q^*(t)$ .

- (a) (5 points) What are the merits of having a variational formulation (such as Hamilton's principle) of classical mechanics when Newton's equation of motion  $F = ma$  is already powerful and all-encompassing?
- (b) (5 points) Please use calculus of variations to show that the true trajectory must satisfy the following "Euler-Lagrange equation"

$$\frac{d}{dt} \left( \frac{\partial L(q^*, \dot{q}^*, t)}{\partial \dot{q}^*} \right) = \frac{\partial L(q^*, \dot{q}^*, t)}{\partial q^*}.$$

- (c) (5 points) If we define the generalized momentum  $p$  associated with  $q$  by

$$p \equiv \frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}}, \quad (1)$$

and it is known that  $\partial L / \partial t = 0$ , please show that the Hamiltonian  $H$  defined by

$$H \equiv p\dot{q} - L = \left( \sum_{j=1}^N p_j \dot{q}_j \right) - L \quad (2)$$

is a constant of the motion.

- (d) (5 points) If we invert Eqn.(1) to write  $\dot{q}$  in terms of  $q, p$  and  $t$ , and substitute it into Eqn.(2) to write  $H = H(q, p, t)$ , then it can be shown (but you are *not* asked to show) that Euler-Lagrange equation implies that the true trajectory  $(q^*, p^*, t)$  in the extended phase space  $(q, p, t)$  must obey the following "Hamilton's equations of motion:"

$$\begin{aligned} \frac{dq^*}{dt} &= \frac{\partial H(q^*, p^*, t)}{\partial p^*}, \\ \frac{dp^*}{dt} &= -\frac{\partial H(q^*, p^*, t)}{\partial q^*}. \end{aligned}$$

Please use calculus of variations to show that the following integral

$$\int_{t=0}^{t=T} p dq - H dt$$

for any trial functions  $(q(t), p(t))$  satisfying  $q(0) = q^*(0)$  and  $q(T) = q^*(T)$  is extremized by  $(q^*, p^*)$  which obeys Hamilton's equations of motion. This variational formulation is yet another version of "Hamilton's principle."

- (e) (10 points) The Hamiltonian for a vertical projectile of unit mass near the surface of the earth is

$$H(q, p) = \frac{p^2}{2} + gq, \quad (3)$$

where  $g > 0$  is the local gravitational acceleration, to be treated as a constant. Danny decides to consider the motion of a particle whose Hamiltonian  $H_2(Q, P, t)$  is defined by

$$\begin{aligned} H_2(Q, P, t) &\equiv \frac{(P - gt)^2}{2} + g \left( Q - \frac{1}{2}gt^2 \right) \\ &= \frac{P^2}{2} + g(Q - Pt). \end{aligned}$$

1. (5 points) Please show that  $(Q - Pt)$  is a constant of the motion.
  2. (5 points) Can we think of  $(Q, P)$  as just another set of generalized coordinate and momentum used to describe that same vertical projectile of Eqn.(3)? You must support your claim with a valid argument or derivation to earn the credits.
- (f) (10 points) If we can find two functions  $Q(q, p, t)$  and  $P(q, p, t)$  and a certain new Hamiltonian  $K(Q, P, t)$  such that

$$pdq - H(q, p, t) dt = PdQ - K(Q, P, t) dt + dF,$$

where  $F$  is some function of  $(q, p, t)$ , then we say that  $(q, p) \mapsto (Q, P)$  forms a canonical transformation. Let us consider once again the vertical projectile of Eqn.(3). Maria decides to adopt and go along with some coordinate system  $Q$  which is freely falling in the earth's gravitational field, that is,

$$Q \equiv q + \frac{1}{2}gt^2.$$

Physically, we know all particles acted on solely by the earth's gravitational pull now appear to be moving like free particles to Maria. So, it is natural for Maria to adopt the new Hamiltonian

$$K(Q, P, t) \equiv \frac{P^2}{2}$$

to describe the motion of the particle of Eqn.(3). Question: Can we find a certain function  $P(q, p, t)$  such that  $(q, p) \mapsto (Q, P)$  forms a canonical transformation? (Explicit derivation or verification of your claim is required.)

3. (25 points) (Hamilton-Jacobi theory.) Let  $q \equiv (q_1, q_2, \dots, q_N)$  be the generalized coordinate and  $p \equiv (p_1, p_2, \dots, p_N)$  be the canonical momentum conjugate to  $q$ , respectively. Suppose a particle with an initial position  $q_0$  at the initial time  $t_0$  moves to a position  $q_f$  at some final time  $t_f$ , and denote the true trajectory connecting these two end

points by  $(q^*(t), p^*(t), t)$  in the extended phase space  $(q, p, t)$ , then we may consider a function  $S$  defined by

$$S(q_f, t_f, q_0, t_0) \equiv \int_{t=t_0}^{t=t_f} p^* dq^* - H(q^*, p^*, t) dt \equiv \int_{t=t_0}^{t=t_f} \left( \sum_{j=1}^N p_j^* dq_j^* \right) - H(q^*, p^*, t) dt, \quad (4)$$

where  $H$  is the Hamiltonian of the system.

(a) (10 points) Please use Hamilton's principle to show that

$$\begin{aligned} p_f &= \frac{\partial S}{\partial q_f}, \\ -p_0 &= \frac{\partial S}{\partial q_0}, \\ -H(q_f, p_f, t_f) &= \frac{\partial S}{\partial t_f}. \end{aligned}$$

Note: Do remember that the *entire* trajectory  $(q^*(t), p^*(t), t)$  changes when you vary either end point!

- (b) (5 points) Suppose you are provided with the exact functional form of  $S$ . Briefly explain how the time dependence  $q(t)$  generically can be derived by just solving a set of algebraic equations (rather than a set of differential equations) when the initial data  $(q_0, p_0)$  at the time  $t_0$  have been specified.
- (c) (10 points) By the results of Part (a), we know  $S$  satisfies the "Hamilton-Jacobi equation:"

$$-H\left(q_f, \frac{\partial S}{\partial q_f}, t_f\right) = \frac{\partial S}{\partial t_f}.$$

For the Hamiltonian of Eqn.(3), we then must solve

$$-\left[\frac{(\partial S/\partial q)^2}{2} + gq\right] = \frac{\partial S}{\partial t}. \quad (5)$$

Using "separation of variables," that is, by assuming that

$$S(q, t) \equiv W_0(t) + W_1(q)$$

for some functions  $W_0$  and  $W_1$ , one can derive a certain type of solutions to Eqn.(5). Please use this trick to find  $S(q, t)$ .