量子力學一　　　　2013 資格考

Each of the problems given below accounts for 20 points:

1. Consider a delta-function potential $V(x) = -V_0\,\delta(x)$, where V_0 is a positive constant. What are the reflection and transmission coefficients for this potential in terms of the energy E of the incident particle?

2. The transition operator for a finite spatial displacement $\vec{a}$ is give by

$$T(\vec{a}) = \mathrm{e}^{-i\vec{p}\cdot\vec{a}/\hbar},$$

where $\vec{p}$ is the momentum operator.

   (a) Evaluate $[x_i, T(\vec{a})]$, where x_i is the i^{th} component of the position operator $\vec{x}$.

   (b) Given that $\vec{x}|\vec{b}\rangle = \vec{b}|\vec{b}\rangle$, show that $T(\vec{a})|\vec{b}\rangle$ is also an eigenstate of the position operator $\vec{x}$. What is the corresponding eigenvalue?

3. A particle moving in one dimension interacts with a potential $V(x)$. If $|\psi\rangle$ is a stationary state of the system, show that

$$\langle\psi|x\frac{dV}{dx}|\psi\rangle = \langle\psi|T|\psi\rangle,$$

where $T = p^2/2m$ is the kinetic energy of the particle.

(Hint: $\dfrac{d}{dt}\langle\psi|xp|\psi\rangle = 0$.)

4. The angular momentum ladder operators $J_\pm$ are defined as

$$J_+ = J_x + iJ_y, \qquad J_- = J_x - iJ_y.$$

Show that

$$J_\pm|jm_j\rangle = C_\pm|j, m \pm 1\rangle,$$

in which $|jm_j\rangle$ are normalized simultaneous eigenstates of $\vec{J}^2$ and J_z. What are C_+ and C_-?

5. Consider a spin-$\frac{1}{2}$ system represented by the normalized state vector

$$\begin{pmatrix} \cos\alpha \\ \sin\alpha\ \mathrm{e}^{i\beta} \end{pmatrix}.$$

What is the probability that a measurement of S_y yields $-\frac{\hbar}{2}$?