Spring 2017

Qualifying Exam — Quantum Mechanics I

1. Consider a two-level system whose Hamiltonian is given by

$$H = i\alpha \left(|\phi_2\rangle \langle\phi_1| - |\phi_1\rangle \langle\phi_2| \right) \ ,$$

where α is a real number having the dimension of energy and $|\phi_1\rangle$ and $|\phi_2\rangle$ are orthonormal state kets. In the following, always express your answers in the above basis.

   (a) (5%) Explain whether H can be a projection operator.

   (b) (5%) Find the eigenstates of H and their corresponding energy eigenvalues E_1 and E_2.

   (c) (10%) If the system is initially (*i.e.*, $t = 0$) in the upper state $|\psi(0)\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, find the probability so that a measurement of energy at $t = 0$ yields: (i) E_1 and (ii) E_2, respectively.

   (d) (10%) Find the state $|\psi(t)\rangle$ at time t.

2. Consider a one-dimensional simple harmonic oscillator of mass m and charge q. Suppose the system is placed in a static electric field of strength E. The Hamiltonian of this oscillator is given by

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2 - qE\hat{x} \ .$$

The ground-state wave function when $E = 0$ is given by

$$\psi_0(x) = \frac{1}{(\sqrt{\pi} x_0)^{1/2}} \exp\left[-\frac{1}{2} \left(\frac{x}{x_0} \right)^2 \right] \ ,$$

where $x_0 \equiv \sqrt{\hbar/(m\omega)}$.

   (a) (10%) For a constant electric field E, find the energy levels for all states.

   (b) (15%) Determine the most likely position of the oscillator in the ground state and give a physical interpretation for your result.

   (c) (15%) Now consider the case where the electric field is an oscillating field, $E = E_0 \cos(\omega' t)$, with ω' different from ω in general. Go to the Heisenberg picture. Compute the following quantities

$$\frac{d\hat{x}}{dt} \ \text{and} \ \frac{d\hat{p}}{dt} \ .$$

3. Suppose we quantize the angular momentum along the z-direction. Consider a spin system with the three $s = 1$ states $|1, +1\rangle$, $|1, 0\rangle$, and $|1, -1\rangle$ denoted by

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

respectively.

(a) (15%) Construct the explicit matrix representations of the spin angular momentum operators $S_{x,y,z}$ and $\mathbf{S}^2$ and show that they are correctly normalized and satisfy the desired Lie algebra.

(b) (15%) Suppose that a state, denoted by $|\alpha\rangle$, has a $+\hbar$ eigenvalue of $S_n \equiv \hat{\mathbf{n}}\cdot\mathbf{S}$ along the direction $\hat{\mathbf{n}} = (\sin\theta, 0, \cos\theta)$. Express this state in terms of the eigenstates of S_z.