NTU-Physics Statistical Physics Qualifying Exam (2017)

(Please note: 5 problems on 2 pages; Answers in both Chinese and English are OK.)

The following formulas may be useful:

(1) $\displaystyle\sum_{\mathbf{k}} \to \frac{L^d}{(2\pi)^d}\int d^d k; \quad \sum_{\mathbf{p}} \to \frac{L^d}{h^d}\int d^d p \quad (V=L^d \to \infty),$ d is the dimensionality of the box.

(2) $\displaystyle I_v \equiv \int_0^\infty e^{-\alpha y^2} y^v dy = \begin{cases} (1/2)\sqrt{\pi/\alpha} & \text{for } v=0 \\ (1/4)\sqrt{\pi/\alpha^3} & \text{for } v=2 \end{cases}$

(3) Bose-Einstein integrals $\displaystyle g_n(z) = \frac{1}{\Gamma(n)}\int_0^\infty \frac{x^{n-1}dx}{z^{-1}e^x -1} = \sum_{k=1}^\infty \frac{z^k}{k^n} \quad (0 \le z \le 1), \quad \Gamma(n+1) = n\Gamma(n)$

If $z=1$ ($\mu=0$), $g_n(1) = \zeta(n)$ $\quad n>1$; if $z \to 1$, $g_n(z) \to \infty$ $\quad n \le 1$.

1. (20 %)

(a) (10 %) Show that the partition function $Q_N(V, T)$ of a classical *extreme* relativistic gas consisting of N monatomic molecules with energy-momentum relationship $\varepsilon = pc$, c being

the speed of light, is given by $\displaystyle Q_N(V,T) = \frac{1}{N!}\left\{ 8\pi V (\frac{kT}{hc})^3 \right\}^N$.

(b) (10 %) Calculate the thermodynamic quantities A, S and U. Check that $PV = U/3$, $U/N = 3kT$, and $\gamma = C_P/C_V = 4/3$.

2. (20 %)

(a) (10 %) Calculate the entropy of a system of N classical three-dimensional harmonic oscillators $[h = (1/2)p^2/m + (1/2)m\omega^2 r^2]$ ("classical Einstein solid"), and show that it diverges at $T \to 0$.

(b) (10 %) Show that quantum considerations "save the situation."

3. (20 %) Consider a system of N dipoles with $J = 1/2$. Each dipole then has two orientations

with the corresponding energies of $\pm\varepsilon = \pm\mu_B H$.

(a) (7 %) Calculate the partition function $Q_N(\beta)$ of the system.

(b) (7 %) Calculate the Helmholtz free energy A, the entropy S and the magnetization M of the system.

(c) (6 %) Evaluate the entropy S and magnetization M in the high-temperature limit. What is the Curie constant $C_{1/2}$.

4. (20 %) Consider an ideal Bose gas confined to a region of area A in *two dimensions*.

(a) (10 %) Express the number of particles in the excited states, N_e, and the number of particles in the ground state, N_0, in terms of z, T, and A.

(b) (10 %) Does the system exhibit Bose-Einstein condensation? If it does, what is the Bose-Einstein condensation temperature T_c? If it does not, why?

5. (20 %) The partition function of the one-dimensional Ising model is given by

$$Q_I(B,T) = \sum_{\sigma_1}\sum_{\sigma_2}\cdots\sum_{\sigma_N}\exp[\beta\sum_{k=1}^{N}(J\sigma_k\sigma_{k+1}+B\sigma_k)].$$ Introduce a 2×2 transfer matrix P such

that $\langle\sigma|P|\sigma'\rangle = \exp\{\beta[J\sigma\sigma'+B(\sigma+\sigma')/2]\}$, where $(\sigma, \sigma' = \pm1)$. One can solve the 1-D

Ising model with this transfer matrix by assuming the periodic boundary condition $\sigma_{N+1} = \sigma_1$.

(a) (8 %) Calculate the partition function $Q_I(B,T)$.

(b) (6 %) Calculate the Helmholtz free energy $A_I(B,T)$.

(c) (6 %) Calculate the magnetization $M_I(B,T)$ and show that for any nonzero temperature, there is no spontaneous magnetization in the one-dimensional Ising model.