Ph.D. Qualifying Exam: Quantum Mechanics (II)

February 2017

Problem 1. (18 points) An isolated hydrogen atom has a hyperfine interaction of the form $A\vec{S}_p \cdot \vec{S}_e$ between the proton spin $\vec{S}_p$ and the electron spin $\vec{S}_e$. The two spins have magnetic moments $\alpha\vec{S}_p$ and $-\beta\vec{S}_e$, and the system is in a uniform magnetic field in the *z*-direction $\vec{B} = B_0\hat{z}$, where α, β and B_0 are some positive constants. Consider only the orbital ground state. **(a) (10 points)** Find the exact eigenvalues of this system and sketch the hyperfine splitting spectrum (i.e., sketch the energy-level diagram) as a function of the magnetic field. **(b) (8 points)** Find the eigenstates associated with each energy level.

Problem 2. (30 points) (a) (3 points) The Stark effect (shift of energy levels by a constant external electric field) in atom is usually observed to be quadratic in the field strength. Explain why. **(b) (3 points)** But for some states of the hydrogen atom, the Stark effect is observed to be linear in the field strength. Explain why. **(c)** Illustrate by making a perturbation calculation of the Stark effect of an electric field $\vec{E} = E_0\hat{z}$ to lowest non-vanishing order for **(i) (10 points)** the ground state (n=1) and **(ii) (10 points)** the first excited states (n=2) of the hydrogen atom, ***neglecting spin and relativistic effects***. Here, $\hat{z}$ is a unit vector in the *z*-direction, E_0 is a constant and n denotes the principal quantum number. **(iii) (4 points)** Draw an energy-level diagram for n=2 which shows the levels before and after application of the electric field, and describe the spectral lines that originate from these levels which can be observed (i.e., allowed electric dipole transitions).

Problem 3. (10 points) (a) (5 points) What is the adiabatic theorem and what is the condition for it to be valid? **(b) (5 points)** What is the Fermi golden rule and what is its use?

Problem 4. (27 points) Consider the scattering of a particle of mass m from a square well potential $V(r) = \begin{cases} -V_0, & r < r_0 \\ 0, & r > r_0 \end{cases}$. **(a) (7 points)** Show that the *s*-wave phase shift is $\delta_0 = -kr_0 + \tan^{-1}\left(\frac{k}{k'}\tan k' r_0\right)$, where *k'* and *k* are the wave numbers inside and outside the well. **(b) (i) (5 points)** Show that at low energies (i.e., $kr_0 \ll 1$), the phase shift

$\delta_0 \approx kr_0\left(\frac{\tan k_0 r_0}{k_0 r_0}-1\right)$, where $k_0=\sqrt{\frac{2mV_0}{\hbar^2}}$, and then find the scattering amplitude $f_0(\theta)$. **(ii) (4 points)** From (i), show also that for $k_0 r_0 \ll 1$, the differential cross section is isotropic and the total cross section is given by $\sigma_{tot}=\left(\frac{16\pi}{9}\right)\frac{m^2 V_0^2 r_0^6}{\hbar^4}$. **(c) (7 points)** Calculate the scattering amplitude $f_B(\theta)$ and the differential cross section using the Born approximation. **(d) (4 points)** At low energies, the condition for the Born approximation to be valid is $(k_0 r_0)^2 \ll 1$. In this case, compare the result of the total cross section obtained from the Born approximation with the result in (b)-(ii).

Problem 5. (15 points) (a) Assume that the interaction Hamiltonian between two identical neutrons (each with mass m) is $V(r)=(\vec{\sigma}_1\cdot\vec{\sigma}_2)V_Y(r)$, where $\vec{\sigma}_1$ and $\vec{\sigma}_2$ are the Pauli spin operators of the projectile and target neutrons, respectively (i.e., $\vec{S}_1=\frac{\hbar}{2}\vec{\sigma}_1, \vec{S}_2=\frac{\hbar}{2}\vec{\sigma}_2$), and r is the magnitude of the relative coordinate between the two identical neutrons. Calculate the differential cross section in the center-of-mass frame **(i) (7 points)** for *unpolarized* neutron-neutron scattering. **(ii) (8 points)** for the initial spin states of the projectile and target neutrons being in states $\left|\downarrow\right\rangle_p=\left|s,m_s\right\rangle_p=\left|\frac{1}{2},-\frac{1}{2}\right\rangle_p$ and $\left|\uparrow\right\rangle_t=\left|\frac{1}{2},\frac{1}{2}\right\rangle_t$, respectively **[suppose the scattering amplitude on the potential $V_Y(r)$ in the center-of-mass frame is given as $f_Y(\theta)$ (i.e., no need to evaluate $f_Y(\theta)$ explicitly), and express your answer in terms of the function $f_Y(\theta)$].**

Useful Information:

First few normalized H-atom eigenfunctions:

$$\begin{cases}\psi_{100}=\left(\frac{1}{\pi a_0^3}\right)^{1/2} e^{-r/a_0}, \\ \psi_{200}=\left(\frac{1}{32\pi a_0^3}\right)^{1/2}\left(2-\frac{r}{a_0}\right)e^{-r/2a_0},\end{cases} \qquad \begin{cases}\psi_{210}=\left(\frac{1}{32\pi a_0^3}\right)^{1/2}\left(\frac{r}{a_0}\right)e^{-r/2a_0}\cos\theta, \\ \psi_{2,1,\pm 1}=\mp\left(\frac{1}{64\pi a_0^3}\right)^{1/2}\left(\frac{r}{a_0}\right)e^{-r/2a_0}\sin\theta e^{\pm i\phi}.\end{cases}$$

Useful Integral: $\int_0^\infty x^n e^{-\alpha x}dx=\frac{n!}{\alpha^{n+1}}$, for $\alpha>0$.

Useful formula: $\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}$, $\quad \begin{aligned}&\tan x = x+x^3/3+\cdots \text{ for } x\ll 1,\\ &\sin x = x-x^3/6+\cdots \text{ for } x\ll 1,\\ &\cos x = 1-x^2/2+\cdots \text{ for } x\ll 1.\end{aligned}$