2021 Ph.D. Qualifying Exam — Quantum Mechanics

1. {20%} A particle of mass m is confined in a one-dimensional region of constant potential at $0 \leq x \leq a$. At $t = 0$ its normalized wave function is

$$\psi(x, t=0) = \sqrt{\frac{8}{5a}} \left[1 + \cos\left(\frac{\pi x}{a}\right)\right] \sin\left(\frac{\pi x}{a}\right).$$

   (a) {10%}What is the wave function at a later time $t = t_0$?

   (b) {10%}What is the probability that the particle is found in the left half of the box (i.e., in the region $0 \leq x \leq a/2$) at $t = t_0$?

2. {20%} A particle of mass m and electric charge q can move only in one dimension and is under the influence of a harmonic force and a homogeneous electric field $\mathcal{E}$. The Hamiltonian of the system is

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 - q\mathcal{E}x.$$

   (a) {10%}Consider the eigenvalue problem

$$H|n\rangle = E_n|n\rangle$$

   with $\psi(x) = \langle x|n\rangle$ being the eigenfunction. What are the possible eigenvalues E_n? [You may make use of the fact that the eigenfunction $\psi(x)$ may be related to the eigenfunction $\psi^0(x)$ of the standard harmonic oscillator Hamiltonian $H_0 = \dfrac{p^2}{2m} + \dfrac{1}{2}m\omega^2 x^2$ by an appropriate shift in the argument x.]

   (b) {10%}Find the operator T that will gives us $|n\rangle$ by acting T on $|n^0\rangle$ where $|n^0\rangle$ is the eigen ket of H^0 with eigenvalue $(n + \frac{1}{2})\hbar\omega$.

3. {30%} Consider the one-dimensional quantum mechanical system described by the Hamiltonian

$$H = \frac{P^2}{2m} - aV_0\,\delta(X)\,,$$

where a and V_0 are positive constants, $\delta(x)$ is the Dirac delta-function, and the position operator X and the momentum operator P satisfy $[X, P] = i\hbar$.

(a) {10%} What are the boundary conditions on the wave functions and their first derivatives at $x = 0$?

(b) {10%} Find the energy eigenvalues and wave functions of the bound states in this singular potential well.

(c) {10%} Evaluate the reflection probability R and the transmission probability T for scattering off the $V(x) = -aV_0\,\delta(x)$.

4. {30%} $\vec{L} = (L_x, L_y, L_z)$ is the angular momentum operator $\vec{L} = \vec{r} \times \vec{p}$.

(a) {5%} Given the canonical commutation relations $[x_i, p_j] = i\hbar\delta_{ij}$, obtain the commutation relations among L_i, $i = x, y, z$.

(b) {5%} Define $L_\pm = L_x \pm iL_y$. What are $[L_z, L_\pm]$?

(c) {5%} Let $|\ell, m\rangle$ be the simultaneous eigenstate of L^2 and L_z with eigenvalues $\ell(\ell+1)\hbar^2$ and $m\hbar$, respectively. Is $L_\pm|\ell, m\rangle$ an eigenstate of L_z? If yes, what is the eigenvalue?

(d) {15%} Consider a system described by the Hamiltonian

$$H = a\vec{L}^2 + bL_z,$$

where a and b are constant parameters. If the initial wave function is given by

$$\langle\theta, \phi|\Psi(t = 0)\rangle = \sqrt{\frac{3}{4\pi}}\sin\theta\sin\phi$$

in the spherical coordinates, what is $\langle\theta, \phi|\Psi(t)\rangle$?

Note that $\langle\theta, \phi|\ell, m\rangle = Y_{l,m}(\theta, \phi)$ are the spherical harmonics. And, in particular,

$$Y_{1,\pm}(\theta, \phi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\phi}, \qquad Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta.$$