NTU Examination of PhD Qualifacation (March 6&7, 2021)
Classical Mechanics

In the following, $\hat{e}_x,\ \hat{e}_y,\ \hat{e}_z$ are the unit vectors along the $x,\ y,\ z-$axes, respectively.

1. A point object of mass m is originally rest on the top of a vertical motionless hoop of radius R. The bottom of the hoop is fixed on the ground. At time $t = 0$ the object starts to fall along the hoop.
   (a) Using the polar coordinate system $(r,\ \theta)$ and introducing the Lagrange undetermined multiplyer, set up the Lagrange equations. (4%)
   (b) Find the height at which the object falls off the hoop. (7%)
   (c) Identify the physical meaning of the undetermined multiplyer. (3%)
2. A particle of mass m and charge q movevs with a velocity $\vec{v} = v\hat{e}_x$, with $v > 0.5c$.
   (a) From the particle's relativistic momentum, show that its free Lagrangian (that is, its Lagrangian when moves without any potential) is $L_{(free)} = -mc^2\sqrt{1-(\frac{v}{c})^2}$. (6%)
   (b) Its Lagrangian becomes $L = L_{(free)} - q\phi + q\vec{A}\cdot\vec{v}$, with the scalar and the vector potentials ϕ and $\vec{A}$ being functions of the position vector $\vec{r}$, as it faces an EM field. Find its Hamiltonian. (6%)
3. Two point particles each of mass m_1 and another one of mass m_2, are constrained to lie on a horizontal circle of radius r. They are mutually connected by springs, each of force constant k, that follow the arc of the circle and they are of equal length when the system is initially in equilibrium and at rest. At time $t = 0$ a small disturbance is given to the system so that the whole system is set to oscillate. Angles $\theta_1,\ \theta_2$,(for particles $1,\ 2$ of the same mass) and θ_3 (for the one of mass m_2) describe their angle displacements from equilibrium. The circle is not deformed and its shape remains unchanged during the oscillation. Neither friction nor air resistant force appears.
   (a) Write down the kinetic and the potential energies of the system and thus set up the equations of motions. (5%)
   (b) Find the normal mode frequecies and cooridinates. (8%)
   (c) Draw figures showing how the system oscillates. (3%)
4. A certain rigid body may be replaced by the three point masses, each of mass m_0, at $(0, 1, 1)$, $(1, 1, 0)$, and $(1, 0, 1)$, here coordinates in the unit of r_0= constant, that is, "1" means "r_0", "2" means "$2r_0$", and so on.
   (a) Find the principal moments of inertia and principal axes. (8%)
   (b) The system rotates with the angular velocity $\omega = \omega_0(\hat{e}_x + \hat{e}_y)$ in the original coordinate system. In the new coordinate system formed by the principal axes, what is the rotation inertia of moment, the angular momentum and the kinetic energy? (6%)

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5. For the system of a certain planet and the Sun with a central force $\vec{f} = -\frac{k}{r^2}\frac{\vec{r}}{r}$, k being a constant, there exists a conserved vector called Laplace-Runge-Lenz vector defined as $\vec{A}_R = \vec{p} \times \vec{L} - mk\frac{\vec{r}}{r}$, with m being the reduced mass of the system, $\vec{r}$ the position vector of the planet to the Sun, and $\vec{p}$, $\vec{L}$ the planet's linear and angular momenta around the Sun.

   (a) Let $\vec{r}_p$, $\vec{r}_a$ be a planet's closest and furthest position vectors to the Sun, respectively. Prove that $\vec{A}_R$ is parallel to either $\vec{r}_p$, or $\vec{r}_a$. (6%)

   (b) From A_R, show that the planet's orbit equation is in the form of $\frac{1}{r} = \frac{1}{r_0}(1 + \varepsilon cos\theta)$. Your final results should have r_0 and ε written in terms of the conserved quantities L, A_R, and constants m, k. Plot a graph of the planet's orbit showing the planet's position by (r, θ) . (6%)

6. An object is moving in a two-dimensional space, with the Hamiltonian being $H = \frac{p_x^2+p_y^2}{2m} + V(x) + \frac{1}{2}ky^2$, where $V(x) = Ax$, $(A > 0)$ as $x >$ 0, and $V(x) = \infty$ as $x \leq 0$.

   (a) Write the Hamilton-Jacobi equations for the characteristic functions $W_x(x)$ and $W_y(y)$, and also write down the expressions for solving them. (You do't have to solve them explicitly.) (5%)

   (b) Express the total mechanic energy E in terms of action variables J_x, J_y. (5%)

   (c) Find the motion frequencies in both directions. (5%)

7. According to the modified Hamilton's principle, a canonical transformation between a system's coordinates $q_i \to Q_i$, momenta $p_i \to P_i$, and Hamiltonians $H \to K$ satisfies that $\sum_i p_i\dot{q}_i - H = \sum_i P_i\dot{Q}_i - K + \frac{dF}{dt}$, with F being any function of coordinates, q_i, Q_i, momenta p_i, P_i, and time t. Among the several possible F's, let us set $F = F_2(q_i, P_i, t) - \sum_i Q_i P_i$ here.

   (a) Prove that $p_i = \frac{\partial F_2}{\partial q_i}$; $Q_i = \frac{\partial F_2}{\partial P_i}$ and $K = H + \frac{\partial F_2}{\partial t}$. (4%)

   (b) Prove that the transformation $Q_k = 2q_k + 3p_k$; $P_k = q_k + 2p_k$ is canonical if $(q_k,\ p_k)$ are canonical by any means. (4%)

   (c) Find the generator $F_2(q_k, P_k)$ for the transformation in part (b). (5%)

   (d) If $K = 0$, prove that F_2 is the action of the system. (4%)