Statistical Physics Qualifying Exam, 2014

1. (30 pts) (A toy model for a one-dimensional rubber band.) A one-dimensional polymer is consisted of N segments of length a linked together end-to-end. The angle between two successive segments can be either $0°$ or $180°$, and there is no change in the energy when a segment flips from one angle to the other. We define the length of the polymer to be $L \equiv na \equiv |N_+ - N_-|a$, where N_+ (N_-)is the number of segments going towards the positive (negative) direction of the x-axis. (Note that $N_+ + N_- = N$ always.) Let $\Omega(N, n)$ be the number of microstates corresponding to the length L.

   (a) (5 pts) Explain why
   $$\Omega(N, n) = 2 \cdot \frac{N!}{\left(\frac{N+n}{2}\right)! \left(\frac{N-n}{2}\right)!}.$$

   (b) (10 pts) Assuming that $N \gg n \gg 1$, please use Boltzmann's formula $S = k_B \ln \Omega$ and the crudest form of Stirling's formula $m! \sim m^m e^{-m}$ to express the entropy S as a function of N and L.

   (c) (10 pts) Using the combined first and second law of thermodynamics
   $$TdS = dU - fdL - \mu dN$$
   to find the *stretching* force f in terms of N, L and the temperature T.

   (d) (5 pts) Find the chemical potential μ in terms of N, L and T.

2. (35 pts) N non-interacting spin-$\frac{1}{2}$ identical quantum particles of mass m are confined to move inside a volume V. Assume the temperature can be approximated as zero, and relativistic correction is unimportant.

   (a) (10 pts) Please find the internal energy U of the system in terms of m, $\hbar$, V and N.

   (b) (8 pts) Using $TdS = dU + PdV$ to compute the degenerate pressure P as a function of the number density $\rho_n \equiv N/V$.

   (c) (6 pts) A so-called "white dwarf" star does not have any nuclear fusion going on inside its core to counteract the inward gravitational pull under its own mass. However, it is believed that the gravitational collapse is avoided due to the degenerate pressure exerted by the (free) electrons comprising the star. If we assume electrical neutrality for the star, many nuclei must also be present in the star. Why is it that astronomers choose to ignore the degenerate pressure of the nuclei and mainly focus on that of the electrons?

   (d) (6 pts) One can show that the sound speed c inside a system is given by
   $$c = \sqrt{\frac{\partial P}{m \partial \rho_n}}.$$
   Show that c increases with ρ_n, *and* give a physical explanation why this is expected. (Note: You should note that this assertion does *not* hold for our atmosphere at room temperature. That's why this question.)

(e) (5 pts) If we consider spin-0 quantum particles at zero temperature instead of the spin-$\frac{1}{2}$ particles considered above, what is the sound speed inside, then?

3. (35 pts) In this problem we use canonical ensemble to compute physical quantities.

   (a) (10 pts) If we assume that classical statistics is valid for a one-dimensional simple harmonic oscillator bathed in a heat reservoir of temperature T, and that its energy is a continuous function of x and p given by

   $$E = \frac{p^2}{2m} + \frac{kx^2}{2},$$

   where x is the displacement and p is the momentum associated with x. Please compute the thermally averaged energy of the oscillator.

   (b) (10 pts) If the oscillator's energy can only assume discrete values

   $$E_n = nh\nu, \tag{1}$$

   where ν is the natural frequency of the oscillator, and $n = 1, 2, \ldots$ is a positive integer. Please compute the thermally averaged energy of the oscillator.

   (c) (5 pts) The result of (b) can reduce to that of (a) when we consider certain limit of the temperature T. What is that limit? Please prove that claim under the limit you proposed.

   (d) (5 pts) If we assume that each of the admissible standing electromagnetic waves of frequency ν inside a cavity kept at a temperature T can be treated as a simple harmonic oscillator of natural frequency ν, and if we also assume that the energy of the oscillators is quantized according to Eqn. (1), then it is possible to derive the famous blackbody radiation formula of Planck's. (You are *not* asked to derive it!) In particular, one can prove that the so-called "ultraviolet catastrophe" does not occur. Specifically, the spectral radiant energy density $I(T, \nu)$ decays to zero very fast as $\nu \longrightarrow \infty$ for a given T. Explain *in words* how this is intimately tied to the energy quantization of the oscillator. (Note that no credits will be granted if you merely quote Planck's formula.)

   (e) (5 pts) If we assume that each of the admissible standing elastic waves of frequency ν inside a crystal at a temperature T can be treated as a simple harmonic oscillator of natural frequency ν, and if we also assume that the energy of the oscillators is quantized according to Eqn. (1), then it is possible to show that the specific heat of the crystal decays to zero very fast as $T \longrightarrow 0$. (Again, you are *not* asked to derive it!) Explain *in words* how this is intimately tied to the energy quantization of the oscillator. (Note that no credits will be granted if you merely quote Einstein's or Debye's formula.)