Ph.D. Qualifying Exam: Quantum Mechanics (II)

February 2019

Problem 1. (20 points) Consider two electrons, each with spin angular momentum $s_i = 1/2$ and orbital angular momentum $l_i = 1$. **(a) (3 points)** What are the possible values of the quantum number L for the total orbital angular momentum $\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2$? **(b) (2 points)** What are the possible values of the quantum number S for the total spin angular momentum $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$? **(c) (3 Points)** Using the results from (a) and (b), find the possible quantum number J for the total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$. **(d) (2 points)** What are the possible values of the quantum number j_1 for the total angular momentum $\mathbf{J}_1 = \mathbf{L}_1 + \mathbf{S}_1$ of electron #1? Same question for electron #2. **(e) (7 points)** Construct the Clebsch-Gordon coefficients to express the total angular momentum basis states $\left| j_1 m_{j_1} \right\rangle$ of electron #1 in terms of the uncoupled basis states $\left| m_{l_1} m_{s_1} \right\rangle \equiv \left| l_1 m_{l_1}, s_1 m_{s_1} \right\rangle$. **(f) (3 Points)** Using the results from (d), find the possible quantum number J for the total angular momentum $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$ and compare your answers to the results in (c).

Problem 2. (20 points) Consider a particle of mass μ constrained to move on a circle of radius a in the x-y plane, and the particle has an electric dipole moment **d** along the radial direction of the circle. **(a) (4 points)** Show that the System Hamiltonian $H = L_z^2 / 2\mu a^2$, where $L_z \to -i\hbar \dfrac{\partial}{\partial \phi}$, and ϕ is the azimuthal angles. **(b) (6 points)** Solve the eigenvalue problem of H (i.e., find the energy eigenvalues and the corresponding eigenfuctions) and interpret the degeneracy. **(c) (10 points)** Suppose that the particle is placed in a weak homogeneous constant electric field **E** pointing along the x-direction, determine the first-order and second-order corrections to the eigenenergy levels of the particle if the interaction Hamiltonian with the electric field is treated as a perturbation.

Problem 3. (22 points) (a) (8 points) Show that the lowest order of the average transition rate of a system subject to a periodic perturbation $H_1(t)=V_1 e^{-i\omega t}$ is given by the Fermi's golden rule. **(b) (14 points)** A particle of mass m and charge q is confined in a one-dimensional harmonic oscillator potential of natural frequency ω. **(i) (4 points)** What are the selection rules governing spontaneous emission from excited states for this system? **(ii) (10 points)** Find the spontaneous emission rate from the first excited state to the ground state of this system.

Problem 4. (18 points) (a) (4 points) Explain or argue why for the case of low wave number k or low kinetic energy $E = \dfrac{\hbar^2 k^2}{2m}$ of an incident particle, the scattering of the incident particle from a spherically symmetric potential $V(r)$ in the partial wave expansion can be restricted to a few low angular momentum l values. **(b) (14 points)** Consider the scattering of a particle of mass m from a hard sphere potential $V(r) = \begin{cases} \infty, & r < r_0 \\ 0, & r > r_0 \end{cases}$. **(i) (8 points)** In the very low-energy limit of $kr_0 \ll 1$, find the expression for the *s-wave* ($l=0$) phase shift. **(ii) (6 points)** Calculate the *s-wave* total cross section σ, then compare your answer with the classical geometric cross section πr_0^2, and finally check the validity of *the optical theorem* $\sigma = \dfrac{4\pi}{k} \operatorname{Im} f(\theta = 0)$ in this case, where $f(\theta)$ is the scattering amplitude.

Problem 5. (20 points) (a) (5 points) Assume that the interaction Hamiltonian between two identical neutrons (each with mass m) is $V(r) = V_Y(r)$ which is spin independent, and r is the magnitude of the relative coordinate between the two identical neutrons. Calculate the differential cross section in the center-of-mass frame for *unpolarized* neutron-neutron scattering. **(b) (15 points)** Assume that the interaction Hamiltonian between two identical neutrons (each with mass m) is $V(r) = (\vec{\sigma}_1 \cdot \vec{\sigma}_2) V_Y(r)$, where $\vec{\sigma}_1$ and $\vec{\sigma}_2$ are the Pauli spin operators of the projectile and target neutrons, respectively (i.e., $\vec{S}_1 = \dfrac{\hbar}{2}\vec{\sigma}_1, \vec{S}_2 = \dfrac{\hbar}{2}\vec{\sigma}_2$). Calculate the differential cross section in the center-of-mass frame **(i) (7 points)** for *unpolarized* neutron-neutron scattering, and **(ii) (8 points)** for the initial spin states of the projectile and target neutrons being in states $\left|\uparrow\right\rangle_p = \left|s_1, m_{s1}\right\rangle_p = \left|\dfrac{1}{2}, \dfrac{1}{2}\right\rangle_p$ and $\left|\downarrow\right\rangle_t = \left|s_2, m_{s2}\right\rangle_t = \left|\dfrac{1}{2}, -\dfrac{1}{2}\right\rangle_t$, respectively. **[Note that for Problem 5, let the scattering amplitude on the potential $V_Y(r)$ for two distinguishable particles in the center-of-mass frame be $f_Y(\theta)$ (i.e., no need to evaluate $f_Y(\theta)$ explicitly), and express your answer in terms of $f_Y(\theta)$].**