Statistical Physics Qualifying Exam (2019)

1. Assuming that the entropy S and the statistical number Ω of a physical system are related through an arbitrary functional form

$$S = f(\Omega),$$

show that the additive character of S and the multiplicative character of Ω *necessarily* require that the function $f(\Omega)$ be of the form $S = k \ln\Omega$. [25 points]

2. Show that the entropy of a system in the grand canonical ensemble can be written as

$$S = -k \sum_{r,s} P_{r,s} \ln P_{r,s},$$

where $P_{r,s}$ is given by $\frac{\exp(-\alpha N_r - \beta E_s)}{\sum_{r,s} \exp(-\alpha N_r - \beta E_s)}$. [25 points]

3. Show that the entropy of an ideal gas in thermal equilibrium is given by the formula

$$S = k \sum_{\epsilon} [\langle n_\epsilon + 1\rangle \ln\langle n_\epsilon + 1\rangle - \langle n_\epsilon\rangle \ln\langle n_\epsilon\rangle]$$

in the case of *bosons* and by the formula

$$S = k \sum_{\epsilon} [-\langle 1 - n_\epsilon\rangle \ln\langle 1 - n_\epsilon\rangle - \langle n_\epsilon\rangle \ln\langle n_\epsilon\rangle]$$

in the case of *fermions*. [25 points]

4. Combining $T < T_c = \frac{h^2}{2\pi mk}\left\{\frac{N}{V\zeta(\frac{3}{2})}\right\}^{2/3}$ and $g_{3/2}(z) = (\lambda^3/v) < 2.612$, and making use of the first two terms of the formula

$$g_\nu(e^{-\alpha}) = \frac{\Gamma(1-\nu)}{\alpha^{1-\nu}} + \sum_{i=0}^{\infty} \frac{(-1)^i}{i!}\zeta(\nu - i)\alpha^i,$$

show that, as T approaches T_c from above, the parameter $\alpha(= -\ln z)$ of the ideal Bose gas assumes the form

$$\alpha \approx \frac{1}{\pi}\left(\frac{3\zeta(3/2)}{4}\right)^2\left(\frac{T - T_c}{T_c}\right)^2.$$

[25 points]