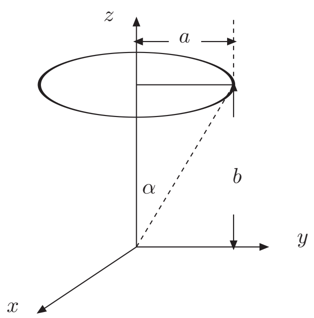
Classical Electrodynamics I

* *All quantities are expressed in terms of SI units.*

* *Vectors are represented with right-pointing arrow notation above their names, as in* $\vec{v}$.

1. [20%] Prove the mean value theorem: For charge-free space the value of the electrostatic potential at any point is equal to the average of the potential over the surface of any sphere centered on that point.

2. A ring of radius a with uniform distribution of charge q is located on the $z = b$ plane and centered at $(0, 0, b)$. The distance from any point on the ring to the origin is $c = \sqrt{a^2 + b^2}$. The potential possesses azimuthal symmetry, therefore $V = V(r, \theta)$.



   (a) [10%] Show that the potential on the positive z axis is

   $$V(r, 0) = \frac{q}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\alpha)$$

   where $\alpha = \tan^{-1}\frac{a}{b}$ and $r_<$ ($r_>$) is the smaller (larger) of r and c.

   (b) [10%] What is the potential $V(r, \theta)$ at any point in space?

3. |

   (a) [20%] For the volume integral of the magnetostatic field $\vec{B}$ inside a sphere of radius R, if all the current density is contained within the sphere, prove that

   $$\int_{r<R} \vec{B} d^3x = \frac{2\mu_0}{3}\vec{m},$$

   where $\vec{m}$ is the total magnetic moment.

   (b) [10%] The magnetic field at a point $\vec{x}$ away from the origin ($\vec{x} \neq 0$) due to a point magnetic dipole $\vec{m}$ located at the origin is given by

   $$\vec{B}_{\vec{m}}(\vec{x}) = \frac{\mu_0}{4\pi} \frac{3(\hat{n}\cdot\vec{m})\hat{n} - \vec{m}}{r^3}$$

   where $r = |\vec{x}|$ and $\hat{n} = \frac{\vec{x}}{r}$. Show that the surface integral for $\vec{B}_{\vec{m}}$ over a spherical surface of radius R and centered at the origin vanishes. *i.e.*,

   $$\oint_{r=R} \vec{B}_{\vec{m}}(\vec{x})\, da = 0$$

   (c) [10%] What is the delta function contribution that must be added to $\vec{B}_m$ for the field of the magnetic dipole in order to include the information of part (a)?

4. The two homogeneous Maxwell equations $\vec{\nabla}\cdot\vec{B} = 0$ and $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ allow us to define the vector potential $\vec{A}$ and scalar potential Φ so that $\vec{B} = \vec{\nabla}\times\vec{A}$ and $\vec{E} = -\vec{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t}$.

   (a) [10%] The potentials $\vec{A}$ and Φ are not uniquely defined. In fact, there is a gauge freedom: we may transform $\left(\vec{A}, \Phi\right)$ to $\left(\vec{A}', \Phi'\right)$ without changing $\vec{E}$ and $\vec{B}$. What is the gauge transformation?

   (b) [10%] The Lorentz gauge is obtained by imposing the condition

   $$\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t} = 0.$$

   What are the other two inhomogeneous Maxwell equations written in terms of the potentials in the Lorentz gauge?