Ph.D. Qualifying Exam: Quantum Mechanics (II)

March 2020

Problem 1. (20 points)

(a) (10 points) For a localized spin-(1/2) electron, find the eigenvectors $|\hat{n},\pm\rangle$ of the operator $\vec{S}\cdot\hat{n}$, where $\vec{S}$ is the spin operators and $\hat{n}=(n_x,n_y,n_z)=(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)$ is a unit vector.

(b) (10 points) Suppose that the electron is in a uniform magnetic field $\vec{\mathbf{B}}_0=B_0\hat{z}$ and at time $t=0$, the electron spin is in the state $|\hat{n},-\rangle$, calculate the expectation value of the spin operator $\langle\vec{S}(t)\rangle$ at time *t*.

Problem 2. (15 points)

A localized spin-(1/2) electron is placed in a magnetic field $\vec{\mathbf{B}}=B_0\,\hat{z}+B_1\cos\omega t\,\hat{x}-B_1\sin\omega t\,\hat{y}$ which is often employed in magnetic resonance experiment. Assume that the particle is in the spin-up state along the $+\hat{z}$ axis at time *t=0*, what is the probability that it will have flipped to $-\hat{z}$ axis at time *t*.

Problem 3. (40 points)

Spin-orbital interactions split *n*=2 degenerate levels of hydrogen atom in two levels *(2s1/2,2p1/2)* and *2p3/2*. Assuming that the spin-orbital Hamiltonian is in the form of $\eta\vec{L}\cdot\vec{S}$, where $\vec{L}$ and $\vec{S}$ are the orbital angular momentum and spin operators, respectively, and η is a constant. Answer the following questions by taking into account the Coulomb interaction and the spin-orbital interaction, but neglecting the relativistic effects.

(a) (6 points) Find the eigenenergies of the electron of the hydrogen atom at principal quantum number *n*=2.

(b) (12 points) Find the total angular momentum *j*-states (eigenstates) in terms of the product states of $|lm_l\rangle$ and $|sm_s\rangle$, where l and s are the values of the orbital and spin angular momenta of the electron, respectively, and $m_l\hbar$ and $m_s\hbar$ are their corresponding z-component values [i.e., find the Clebsch-Gordon (C-G) coefficients for the states of the total angular momentum $\vec{J}=\vec{L}+\vec{S}$.].

(c) (4 points) Express all of the eigenfunctions (eigenstates) of the electron in **(b)** with principal quantum number *n=2* in the spherical coordinate and spinor representations [you can express your answers in terms of $R_{nl}(r)$ and $Y_{lm}(\theta,\phi)$].

(d) (18 point) Consider the response of the *n=2* energy levels of the hydrogen atom to a weak constant electric field $\vec{E}=\varepsilon\,\hat{z}$ (Stark effect) taking into account the energy splitting due to this spin-orbital interaction, where $\hat{z}$ is a unit vector in the *z*-direction, and ε is a constant. That is, by treating the electric dipole Hamiltonian $H_1=-(-e\vec{r})\cdot\vec{E}$ as a perturbation, find the lowest non-zero corrections (if any) to the eigenenergies and their corresponding perturbed states within the *n*=2 subspace.

Problem 4. (25 points)

(a) (9 points) Calculate in the Born approximation the scattering amplitude $f_G(\theta)$ of an incident particle of mass *m* and kinetic energy $E = \hbar^2 k^2 / 2m$ on the potential $V_G(r) = V_0 e^{-ar}$.

(b) (3 points) Calculate the differential cross section dσ/dΩ of the scattering problem in **(a).**

(c) (6 points) Assume that the interaction Hamiltonian between two identical spin-(1/2) neutrons (each with mass m) is $V(r) = (\vec{\sigma}_p \cdot \vec{\sigma}_t) V_G(r)$, where $\vec{\sigma}_1$ and $\vec{\sigma}_2$ are the Pauli spin operators of the projectile and target neutrons, respectively (i.e., $\vec{S}_p = \frac{\hbar}{2}\vec{\sigma}_p, \vec{S}_t = \frac{\hbar}{2}\vec{\sigma}_t$), $V_G(r)$ is defined by the scattering problem in **(a)** but with *r* being the magnitude of the relative displacement between the two identical neutrons. Calculate the differential cross section for *unpolarized* neutron-neutron scattering in the center-of-mass frame **[express the answer in (c) and also the answer in (d) in terms of the scattering amplitude $f_G(\theta)$ defined in (a), i.e., no need to evaluate it explicitly]**.

(d) (7 points) Compute the differential cross section of the scattering problem in **(c)** in the center-of-mass frame for the initial spin states of the projectile and target being in states $\left|\downarrow\right\rangle_p = \left|s, m_s\right\rangle_p = \left|\frac{1}{2}, -\frac{1}{2}\right\rangle_p$ and $\left|\uparrow\right\rangle_t = \left|\frac{1}{2}, \frac{1}{2}\right\rangle_t$, respectively.

First few spherical harmonics:

$$Y_{0,0}(\theta,\phi) = \sqrt{\frac{1}{4\pi}} , \qquad Y_{2,0}(\theta,\phi) = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1) ,$$

$$Y_{1,\pm1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}} e^{\pm i\phi}\sin\theta , \qquad Y_{2,\pm1}(\theta,\phi) = \mp\sqrt{\frac{15}{8\pi}} e^{\pm i\phi}\sin\theta\,\cos\theta,$$

$$Y_{1,0}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta , \qquad Y_{2,\pm2}(\theta,\phi) = \sqrt{\frac{15}{32\pi}} e^{\pm 2i\phi}\sin^2\theta .$$

First few radial functions of hydrogen atom:

$$R_{10}(r) = \left(\frac{4}{a_0^3}\right)^{1/2} e^{-r/a_0},$$

$$R_{20}(r) = \left(\frac{1}{8a_0^3}\right)^{1/2}\left(2 - \frac{r}{a_0}\right) e^{-r/2a_0},$$

$$R_{21}(r) = \left(\frac{1}{24a_0^3}\right)^{1/2} \frac{r}{a_0} e^{-r/2a_0}.$$

Useful Integral: $\int_0^\infty x^n e^{-\alpha x} dx = \frac{n!}{\alpha^{n+1}}$, for $\alpha > 0$.