1. (a) For a free, non-relativistic particle confined to a cubic box of side L ($V = L^3$), we have $\varepsilon(n_x, n_y, n_z) = \frac{h^2}{8mL^2}(n_x^2 + n_y^2 + n_z^2)$; $n_x, n_y, n_z = 1, 2, 3\ldots$. The number of distinct microstates for a particle of energy ε is denoted by $\Omega(1, \varepsilon, V)$. For a system with N such particles, one can have $\Omega(N, E, V)$, where E is the energy of the system. Please show that in a reversible adiabatic process, one gets $PV^{5/3}$ = constant.(**10 pts**)
(b) By evaluating the "volume" of the relevant region of its phase space, show that the number of microstates available to a rigid rotator with angular momentum $\leq J$ is $(J/\hbar)^2$. (**10 pts**)

2. (a) In the canonical ensemble, one has $Q_N(V, T) = \sum_r \exp(-\beta E_r)$, where Q_N is the partition function of a system. Please show that the Helmholtz free energy $A = -kT \ln Q_N(V, T)$. (**10 pts**)
(b) We have a system of N independent harmonic oscillators. Please calculate the Helmholtz free energy A and the energy U of the oscillators by treating the oscillators quantum mechanically. (**12 pts**)

3. The potential energy of 1-dimensional, *anharmonic* **classical** oscillator may be written as $V(q) = cq^2 - fq^3 - gq^4$, where c, f and g are positive constants and f and g are very small values. What is the leading contribution of anharmonic terms to (a) the heat capacity (**10 pts**) and (b) the mean value of the position coordinate q? (**10 pts**)

4. (a) Consider a classical ideal gas of N particles. Please write down the grand partition function of the gas **and** show that $PV = NkT$. (hint: the canonical partition function $Q_N(V, T) = Q_1(V, T)^N/N!$, and you can assume that $Q_1(V, T) = Vf(T)$). (**12 pts**)
(b) The expectation value of a physical quantity G, which is dynamically represented by an operator $\hat{G}$: $\langle G \rangle = \frac{1}{N}\sum_{k=1}^{N} \int \psi^{k*}\, \hat{G}\, \psi^k \, d\tau$. Where the wave functions ψ^k are normalized. One can show that $\langle G \rangle = \mathrm{Tr}(\hat{\rho}\hat{G})$. ($\hat{\rho}$ is the density operator) (You **do not** need to show it)
Consider a single electron placed in a magnetic field $\boldsymbol{B}$. The electron possesses an intrinsic spin $\frac{1}{2}\hbar\hat{\sigma}$ and a magnetic moment μ_B, where $\hat{\sigma}$ is the Pauli spin operator and $\mu_B = e\hbar/2mc$. Let the applied field be in the direction of the z-axis. Please write down the density matrix in the canonical ensemble and obtain the expectation value of σ_z. (**13 pts**)

5. The grand partition function ($z = e^{\mu/kT}$)

$$\mathcal{Q}(z, V, T) = \begin{cases} \prod_{\varepsilon} \frac{1}{(1 - z e^{-\beta\varepsilon})} \; in\ the\ B.E.case\ with\ z e^{-\beta\delta} < 1 \\ \prod_{\varepsilon} (1 + z e^{-\beta\varepsilon}) \; in\ the\ F.D.case \end{cases}$$

Please derive the value of the mean occupation number $\langle n_\varepsilon \rangle$ of level ε for both B. E. and F. D. cases. (**13 pts**)