Quantum Mechanics — Qualifying Exam — 2020

1. {25%} Consider a free particle of mass m moving on a line (1-dimensional space). The initial wave function $\Psi(x, 0)$ at time $t = 0$ is given by

$$\Psi(x, 0) = \delta(x),$$

where $\delta(x)$ is the Dirac delta-function. Use non-relativistic quantum mechanics to find te wave function $\Psi(x, t)$ at time $t > 0$.

Note: $\displaystyle\int_{-\infty}^{\infty} e^{-i(\alpha x^2+\beta x)}dx = \sqrt{\frac{\pi}{i\alpha}}e^{\frac{i\beta^2}{4\alpha}}$ for $\alpha, \beta \in \mathbb{R}$.

2. {25%} Consider a 1-dimensional simple harmonic oscillator for which the Hamiltonian H is given by

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2 \quad \text{with } [X, P] = i\hbar.$$

The annihilation operator a and creation operator $a^\dagger$ are define by

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(X + \frac{i}{m\omega}P\right), \quad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(X - \frac{i}{m\omega}P\right).$$

It is known that the coherent state given by

$$|\phi_\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

is an eigenstate of a. Namely, $a|\phi_\alpha\rangle = \alpha|\phi_\alpha\rangle$, $\alpha \in \mathbb{C}$ and $H|n\rangle = (n + \frac{1}{2})\hbar\omega|n\rangle$.

(a) If initially the oscillator is in the state $|\phi_\alpha\rangle$, i.e., $|\Psi(t = 0)\rangle = |\phi_\alpha\rangle$, what is $|\Psi(t)\rangle$ at time t?.

(b) What are $\langle\Psi(t)|X|\Psi(t)\rangle$ and $\langle\Psi(t)|P|\Psi(t)\rangle$?

3. {25%} An electron is initially in a state represented by the spinor (spin wave function)

$$\Psi(t=0) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

where $\alpha, \beta \in \mathbb{R}$ and $\alpha^2 + \beta^2 = 1$. As usual, the $\pm\hbar/2$ eigenstates of the spin operator $S_z = \hbar\sigma_z/2$ are represented respectively by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Now, suppose that a magnetic field $\vec{B} = B_0\,\hat{y}$ in the y-direction is turned on at $t = 0$. Find the spin wave function of the electron $\Psi(t)$ at time $t > 0$.

Note: The Hamiltonian H is $H = -\vec{\mu}\cdot\vec{B} = \mu_B\,\vec{\sigma}\cdot\vec{B}$, where μ_B is the Bohr magneton, and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ with σ_i $(i = x, y, z)$ being the Pauli matrices.

4. {25%} Consider a spinless particle represented by the wave function

$$\psi = K(x + y + 2z)\,e^{-\alpha r},$$

where $r = \sqrt{x^2 + y^2 + z^2}$, and K and α are real constants.

(a) What is the total angular momentum, $\sqrt{\langle \vec{L}^2 \rangle}$, of the particle?

(b) What is the expectation value of the z-component of angular momentum?

(c) If the z-component of angular momentum, L_z, were measured, what is the probability that the result would be $L_z = +\hbar$?

You may find the following expression for the first few spherical harmonics useful:

$$Y_0^0 = \frac{1}{\sqrt{4\pi}}, \quad Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}}\,\mathrm{e}^{\pm i\phi}\sin\theta$$

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta, \quad Y_2^{\pm 1} = \mp\sqrt{\frac{15}{8\pi}}\,\mathrm{e}^{\pm i\phi}\sin\theta\cos\theta$$