Classical Mechanics

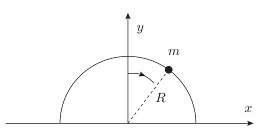
1. [10%] For a system of particles, if we choose a generalized coordinate q_j such that q_j corresponds to the angle of a rotation of all the particles around some axis $\hat{n}$ with $\delta\vec{r}_i = (\hat{n} \times \vec{r}_i)\,\delta q_j$. Show that $Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$, the generalized force corresponding to q_j, is the applied torque along the direction $\hat{n}$ and the generalized momentum corresponding to q_j is the component of the angular momentum along $\hat{n}$.

2. [10%] For the torque-free motion of a rigid body with one point fixed and with the body axes being the principal axes, derive the Euler's equations of motion

$$
\begin{aligned}
I_1\dot{\omega}_1 - \omega_2\omega_3\left(I_2 - I_3\right) = 0 \\
I_2\dot{\omega}_2 - \omega_3\omega_1\left(I_3 - I_1\right) = 0 \\
I_3\dot{\omega}_3 - \omega_1\omega_2\left(I_1 - I_2\right) = 0
\end{aligned}
$$

where $\omega_1, \omega_2, \omega_3$ are the components of angular velocity relative to the body axes.

3. [20%] A particle of mass m slides without friction from the top of a fixed hemisphere of radius R as shown in the figure.



The only external force is that of gravity. Use the method of Lagrange multipliers to find the point at which the particle falls off the hemisphere.

4. [20%] Assume the earth of mass m deviates slightly from a sphere, being closely approximated by an oblate spheroid of revolution with $I_2 = I_1$ and z-axis being the symmetry axis of the earth.Choose the origin to be center of mass. For a point $\vec{x}$ located outside the earth, show that the gravitational potential generated by the earth may be approximated by

$$
\begin{aligned}
V\left(r,\theta,\phi\right) &= -G\int d^3x'\,\frac{\rho\left(\vec{x}'\right)}{\left|\vec{x}-\vec{x}'\right|} \\
&\simeq -\frac{Gm}{r} + \frac{G\left(I_3 - I_1\right)}{r^3}P_2\left(\cos\theta\right)
\end{aligned}
$$

where (r,θ,ϕ) is the spherical coordinates of the observation point $\vec{x}$ and $P_2\left(\cos\theta\right) = \frac{3\cos^2\theta - 1}{2}$.

5. [20%] If $f\left(q,p,t\right)$ and $g\left(q,p,t\right)$ are explicitly time dependent and are two constants of motion $\frac{df(q,p,t)}{dt} = \frac{dg(q,p,t)}{dt} = 0$ where (q,p) are canonical variables, show that the Poisson $\left[f\left(q,p,t\right), g\left(q,p,t\right)\right] = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}$ is itself a constant of motion, *i.e.*, $\frac{d}{dt}\left[f\left(q,p,t\right), g\left(q,p,t\right)\right] = 0$. You may utilize Jacobi identity without providing the proof for its validity.

6. [20%] The Lagrangian for a linear triatomic molecule is

$$
\begin{aligned}
L = \frac{m}{2}\left(\dot{x}_1^2 + \dot{x}_3^2\right) + \frac{M}{2}\dot{x}_2^2 - \frac{k}{2}\left(x_2 - x_1 - b\right)^2 \\
- \frac{k}{2}\left(x_3 - x_2 - b\right)^2
\end{aligned}
$$

Introduce the internal coordinates $y_1 = x_2 - x_1, y_2 = x_3 - x_2$, and eliminate x_2 by requiring that the center of mass remain at the origin $x_{cm} = 0$. Obtain the frequencies of the normal modes in y_1 and y_2 coordinates.