Classical Electrodynamics (I) PhD Qualifying Exam (4 problems)

Note: (1) This is a closed-book exam. Notes, dictionary, calculator, and cell phone are NOT allowed.
(2) No one can sit side-by-side with you.
(3) Terms and notations follow the textbook of J. D. Jackson, if not mentioned additionally.
(4) If answers have units different from the SI, please describe it explicitly.

1. Electrostatics

(a) An infinitely large plate with a surface charge density of $+\sigma$ is put on the x-y plane and has infinitesimal thickness. A second infinitely large plate with infinitesimal thickness has a surface charge density of $-\sigma$ and is positioned parallel to the x-y plane at $z = d$. Write down the electric field $\boldsymbol{E}$ in all space. (10%)

(b) The three-dimensional space with $x \geq 0$ is occupied by a grounded, ideal conductor. On the x-axis there is a point charge q at $x = -d$ with $d > 0$ and another charge $-q$ at $x = -2d$. Write down $\boldsymbol{E}$ in all space. (10%)

(c) An isolated, ideally conducting sphere with a radius of $2a$ is centered at the origin. On the sphere there is the total charge of Q. A point charge of q is located on the y-axis at $y = 4a$. Write down $\boldsymbol{E}$ in all space. (10%)

2. Electric dipole and dielectrics

(a) A point charge $-q$ is positioned at $x = d$ and another point charge q at $x = -d$. Write down the electrostatic potential Φ in all space to the first order of d in the limiting case with $d \to 0$ while qd remains finite. (6%)

(b) A linear, isotropic dielectric having an electric permittivity ϵ fills the three-dimensional space with $y \geq 0$. In the region of $y < 0$ there is a uniform electric field $\boldsymbol{E} = 2E_0\, \boldsymbol{y} + E_0\, \boldsymbol{z}$. Here $\boldsymbol{y}$ and $\boldsymbol{z}$ are the unit vectors along the y- and z-axis. On the plane at $y = 0$ there is no free charge. Write down $\boldsymbol{E}$ in the dielectric and derive the induced surface charge density σ on the plane at $y = 0$. (14%)

3. Magnetostatics

(a) There is a vector field $\boldsymbol{A} = J_0\, \boldsymbol{x} - x^2 J_0\, \boldsymbol{z}$ with J_0 as a constant. $\boldsymbol{x}$ and $\boldsymbol{z}$ are the unit vectors along the x- and z-axis. Derive the magnetic field $\boldsymbol{B}$ and the current density $\boldsymbol{J}$. (10%)

(b) The dielectric in the question 2(b) is replaced by a linear, isotropic magnetic material with a magnetic permeability μ, and $\boldsymbol{E}$ in the region of $y < 0$ is replaced by the uniform magnetic field $\boldsymbol{B} = B_0\, \boldsymbol{y} + 3B_0\, \boldsymbol{z}$. There is no free current at $y = 0$. Write down $\boldsymbol{B}$ in the magnetic material. (10%)

4. Maxwell equations

(a) Write down the Maxwell equations for the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$ in linear, isotropic medium without free charge and current densities in differential form. (12%)

(b) Based on the question 4(a), derive the wave equations of $\boldsymbol{E}$ and $\boldsymbol{B}$. (8%)

(c) Write down the speed of light and the index of refraction in the medium according to the question 4(b). (6%)

(d) According to the question 4(c), write down the wave equations for the vector potential $\boldsymbol{A}$. (4%)