Quantum Mechanics II: PhD Qualifying Examination 2023

Instruction: Each question carries equal 25 marks.

1. A beam of particles of mass m and momentum $p = \hbar k$ is incident along the z-axis. The beam scatters off a spherically symmetric potential $V(r)$.

   - Write down the asymptotic form of the wave function in terms of the scattering amplitude $f(\theta)$, where θ is the scattering angle, draw a diagram to label your coordinates explicitly.
   - The incoming plane wave and the scattering amplitude can be expanded in partial waves as:

   $$e^{ikr\cos\theta} \sim \frac{1}{2ikr}\sum_{l=0}^{\infty}(2l+1)\left(e^{ikr} - (-1)^l e^{-ikr}\right)P_l(\cos\theta), \tag{1}$$

   $$f(\theta) = \sum_{l=0}^{\infty}\frac{2l+1}{k}f_l P_l(\cos\theta), \tag{2}$$

   where $P_l(\cos\theta)$ is Legendre polynomial. Define S-matrix from this, and assume it is unitary to show that the coefficient:

   $$f_l = e^{i\delta_l}\sin\delta_l \tag{3}$$

   for some real phase δ_l.

   - Finally, obtain an expression for the total cross-section σ_T in terms of the phase shifts δ_l.
   You may use:

   $$\int_{-1}^{1} dw P_l(w)P_{l'}(w) = \frac{2}{2l+1}\delta_{ll'}. \tag{4}$$

2. A particle of mass m is confined by the infinite square well potential of width $2a$:

   $$V(x) = \begin{cases} 0, & |x| < a, \\ \infty, & |x| > a. \end{cases} \tag{5}$$

   The potential is then perturbed by linear potential $\delta V(x) = \lambda\frac{x}{a}$, $|\lambda| \ll 1$.

   - Show that the energy levels of this particle are unchanged to first order in λ.
   - Show that the ground state wave function is however changed into:

   $$\psi_\lambda(x) = \frac{1}{\sqrt{a}}\cos(\frac{\pi x}{2a}) + \frac{16\lambda}{\pi^2 E_1\sqrt{a}}\sum_{n=2,4,6,\ldots}(-1)^{\frac{n}{2}}\frac{n}{(n^2-1)^3}\sin(\frac{n\pi}{2a}x) + \mathcal{O}(\lambda^2), \tag{6}$$

   where E_1 is the ground state energy. Show and explain why this wave function no longer have well-defined parity.

- Write down the order λ^2 correction to the ground state energy without detailed calculations. Is this positive or negative?

3. Starting with the general Hamiltonian: $H = H_0 + V$, where H_0 is the non-interacting Hamiltonian with eigen state Φ_α and eigen-energy E_α, and V is the interacting potential. Define the incoming Ψ_α^+ and outgoing Ψ_α^- state via Lippmann-Scwhinger equation

$$\Psi_\alpha^\pm = \Phi_\alpha + \frac{1}{(E_\alpha - H_0) \pm i\varepsilon} V \Psi_\alpha^\pm \tag{7}$$

where ε is an infinitesimal positive quantity, and the matrix elements:

$$A_{\beta\alpha}^\pm = \left(\Phi_\beta, V\Psi_\alpha^\pm\right). \tag{8}$$

- Show that $A_{\beta\alpha}^\pm = (A_{\alpha\beta}^\pm)^*$ for $E_\alpha = E_\beta$.
- If V is a separable interaction, i.e. $(\Phi_\beta, V\Phi_\alpha) = f^*(\beta)f(\alpha)$, solve the Lippmann-Schwinger equation exactly to obtain $\Psi_\alpha^\pm$ in this case.
- Given the general S-matrix element is defined to be:

$$S_{\beta\alpha} = \left(\Psi_\beta^-, \Psi_\alpha^+\right) = \delta(\beta - \alpha) + \delta^{(3)}(\vec{p}_\alpha - \vec{p}_\beta)\delta(E_\alpha - E_\beta)M_{\beta\alpha}, \tag{9}$$

  show that $S_{\beta\alpha}$ is unitary, then use this to derive general optical theorem for scattering process.
- Finally using the general optical theorem you derived or otherwise, obtain an explicit expression for the total reaction rate Γ_α of the initial state α, for the separable interaction discussed earlier.

4. The Rayleigh-Ritz quotient for a wave function ψ of a quantum mechanical system with Hamiltonian $\hat{H}$ is:

$$R[\psi] = \frac{(\psi, \hat{H}\psi)}{(\psi, \psi)} \tag{10}$$

where (,) denotes appropriate norm.

- Given that the spectrum of $\hat{H}$ is discrete and that there is a unique ground state of energy E_0, show that $R[\psi] > E_0$ and that equality holds if and only if ψ is the ground state wave function.
- Consider an one-dimensional simple harmonic oscillator (SHO) of mass m and potential:

$$V(x) = \frac{1}{2}m\omega^2 x^2 \tag{11}$$

  Estimate its ground state energy E_0 using the ground state wave funciton of an infinite potential well of width a, centering at origin $x = 0$. You should take a as the variational parameter.
- Estimate similarly the first excited state energy E_1 of SHO by using the first excited state of the infinite potential well as a trial wave function Is E_1 necessarily an upper bound? Explain your answer.

Useful Integrals:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\, x^2 \cos^2 x = \frac{\pi}{4}\left(\frac{\pi^2}{6} - 1\right), \quad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\, x^2 \sin^2 x = \pi\left(\frac{\pi^2}{3} - \frac{1}{2}\right). \tag{12}$$