NTU Examination of PhD Qualifacation (February 24-25, 2024)
Classical Mechanics

In the following, $\hat{e}_i$ is ,the unit vectors along the $i-$axes, and $\dot{a} = \frac{da}{dt}$

1. For the system of a certain planet and the Sun with a central force $\vec{f} = -\frac{k}{r^2}\frac{\vec{r}}{r}$, k being a constant, there exists a conserved vector called Laplace-Runge-Lenz vector defined as $\vec{A}_R = \vec{p} \times \vec{L} - mk\frac{\vec{r}}{r}$, with m being the reduced mass of the system, $\vec{r}$ the position vector of the planet to the Sun, and $\vec{p},\ \vec{L}$ the planet's linear and angular momenta around the Sun.
   (a) Show that $\vec{A}_R$ is either parallel or anti-parallel to the major axis of the planet's orbit around the Sun . Hence it can be used to define the planet's orbit equation. (7%)
   (b) Show that $A_R^2 = 2mL^2E + m^2k^2$, with E being the total (mechanical) energy of the system. (7%)

2. (a) A particle of mass m and charge q movevs with a velocity $\vec{v} = v\hat{e}_x$, with $v > 0.5c$. Its free Lagrangian (that is, its Lagrangian when moves without any potential) is known to be $L_{(free)} = -mc^2\sqrt{1 - (\frac{v}{c})^2}$. As it faces an EM field, its Lagrangian becomes $L = L_{(free)} - q\phi + q\vec{A}\cdot\vec{v}$, with the scalar and the vector potentials ϕ and $\vec{A}$ being functions of the position vector $\vec{r}$. Find its Hamiltonian. (7%)
   (b) In a laboratory, a particle of mass m_0 with momentum $3m_0c$ makes an collision with particle B of mass $2m_0$ initially at rest. No other force is imposed on the system. Find the speed of the center of mass in the laboratory system. (6%)

3. A point object of mass m is originally rest on the top of a vertical motionless hoop of radius R. The bottom of the hoop is fixed on the ground. At time $t = 0$ the object starts to fall along the hoop.
   (a) Using the polar coordinate system $(r,\ \theta)$ and introducing the Lagrange undetermined multiplier, set up the Lagrange equations. (4%)
   (b) Find the height at which the object falls off the hoop. (7%)
   (c) Identify the physical meaning of the undetermined multiplier. (3%)

4. Two objects each of mass m are connected by a special setup of springs so that the potential energy in system is $V(q_1, q_2) = k(2q_1^2 - q_1q_2 + 2q_2^2)$, with $q_i,\ i = 1,2$ being the objects' displacements from their equilibrium positions, and k a positive constant.
   (a) Find A and a so that the transformations $p_1 = A(P_1 + aP_2),\ p_2 = A(P_1 - aP_2);\ q_1 = A(Q_1 + aQ_2),\ q_2 = A(Q_1 - aQ_2)$ being canonical. Show that the transformations can separate variables completely in the Hamilton's characteristic functions $W_i(Q_i),\ i = 1,2$. Write $W_i(Q_i),\ i = 1,2$. (7%)
   (b) From the actions variables find the normal mode frequencies and the normal mode coordinates, then find q_1, q_2. (7%)

— To be continued in next page —

5. (a) The Earth spins daily at the angular velocity $\vec{\omega} = \omega \hat{e}_z$, where $\hat{e}_z$ is along the South-North Poles and pointing to north. Now rotate the coordinate system about y-axis so that the new z-axis is from the center of the Earth and vertically upward through a location with the latitude of $37°N$, $sin\ 37° \approx 0.6$. The new xy-plane is moved upward on the Earth's surface so that the new origin is on the ground of the location. Find the Earth's spin angular velocity as observed in this new coordinate system. (5%)

   (b) A bug of mass m is sitting at $(a, 0, 0)$ on a very big horizontal disk of radius R_0, $R_0 \gg a$. The disk is originally at rest. At time $t = 0$, the disk starts to spin with a constant angular speed ω about the axis perpendicular to the disk through its center(that is, the $z-$axis). Meanwhile, the bug starts to move with a velocity $v_0 \hat{e}_y$, with v_0 being a constant. The experiment is so designed that the centrifugal force caused by the spin can be ignored. (The coordinate system is attached to the disk.)

      i. Find the force due to the Coriolis effect on the bug. (5%)

      ii. Explain why the bug performs a circular motion on the disk. Find the period of this circular motion. (5%)

6. From the modified Hamilton's principle, a canonical transformation between a system's coordinates $q_i \to Q_i$, momenta $p_i \to P_i$, and Hamiltonians $H \to K$ satisfies that $\sum\limits_i p_i \dot{q}_i - H = \sum\limits_i P_i \dot{Q}_i - K + \frac{dF}{dt}$, F being any function of $q_i,\ Q_i$, $p_i,\ P_i$, and time t. Among the several possible F's, let us set $F = F_1(q_i, Q_i, t)$.

   (a) Prove that $p_i = \frac{\partial F_1}{\partial q_i}$; $P_i = -\frac{\partial F_1}{\partial Q_i}$ and $K = H + \frac{\partial F_1}{\partial t}$. (4%)

   (b) Prove that the transformation $Q_k = 2q_k + p_k;\ \ P_k = q_k + p_k$ is canonical if $(q_k,\ p_k)$ are canonical by any means. (4%)

   (c) Find the generator $F_1(q_k, Q_k)$ for the transformation in part (b). (5%)

   (d) If $K = 0$, prove that F_1 is the action of the system. (3%)

7. Two identical pendulums are each made of a massless string of length l with one end fixed to the ceiling apart a distance of L to each other, and the other end attached to a small object of mass m. These two objects are connected by a massless spring of force constant k. The spring is initially in equilibrium so that each string is hung vertically to, and the spring parallel to the ceiling. Now make a small disturbance to one of the mass so that the spring starts a small vibration. A moment later the left (right) string makes an angle θ_1 (θ_2) to the vertical line through the ceiling, and the left (right) object moves a small almost horizontal distance η_1 (η_2) from the initial position when the system is at rest.

   (a) Set up the Lagrange equations for $\eta_1,\ \eta_2$. Find the normal mode frequencies and normal mode coordinates of the vibration. (8%)

   (b) At time $t = 0$, $\eta_1 = A,\ \ \eta_2 = 2A$, and $\dot{\eta}_1 = \dot{\eta}_2 = 0$, find η_1 and η_2. (6%)