Statistical Physics I Qualifying Examination (2024)

Please note: 4 problems on 1 page. Answers in Chinese, English, or Japanese are acceptable.

1 **(15 pts)** For an ideal gas undergoing quasi-static expansion (compression), it is well-known that

$$PV^{\gamma} = \text{constant}$$

where γ has to be determined from the properties of the gas. Consider the loss of translational energy suffered by the molecules of a gas on reflection from a receding wall, derive, for a quasistatic (low wall-receding speed) adiabatic expansion of an ideal nonrelativistic gas, that ***the above relation holds***, with

$$\gamma = \frac{3a+2}{3a}$$

where a is the ratio of the total energy to the translational energy of the gas, that is, $E = aK$.

2 **(15 pts)** Show that the chemical potential of photon gas is equal to zero.

3 **(20 pts)** Give the coordinate representation of the density matrix

$$\hat{\rho} = \exp\left(-\beta\hat{\mathscr{H}}\right)$$

for a *single free* particle of mass m $\left(\hat{\mathscr{H}} = \hat{\mathbf{p}}^2/2m\right)$ in a cubic box with sides L. Calculate the limiting value for $L \to \infty$, using plane-wave solutions for three independent directions.
You may need the integral

$$\int_{-\infty}^{+\infty} \exp\left(-\frac{ax^2}{2} + ixy\right) \mathrm{d}\, x = \sqrt{\frac{2\pi}{a}} \exp\left(-\frac{y^2}{2a}\right)$$

4 In the treatment of high-energy fermions of spin 1/2, one must consider relativistic effects. The single-fermion energy is given by

$$\varepsilon = c\sqrt{p^2 + m^2c^2}$$

(a) **(10 pts)** Assume that the volume V of the system is very large. Show that *the number of states* having the magnitude of p within $|p|$ and $|p| + \mathrm{d}\, p$ is given by (including the spin weight $g = 2$)

$$2\frac{V}{h^3}4\pi p^2 \,\mathrm{d}\, p = \frac{8\pi V}{h^3}(mc)^3 \sinh^2\theta \cosh\theta \,\mathrm{d}\,\theta$$

Show that the average values of the totoal number N, the total energy, and the pressure-volume product are given by the expressions:

(b) **(10 pts)** $N = \dfrac{8\pi V m^3 c^3}{h^3} \displaystyle\int_0^{\infty} \frac{\sinh^2\theta \cosh\theta \,\mathrm{d}\,\theta}{\exp(\beta\,[mc^2\cosh\theta - \mu]) + 1}$

(c) **(10 pts)** $E = \dfrac{8\pi V m^4 c^5}{h^3} \displaystyle\int_0^{\infty} \frac{\sinh^2\theta \cosh^2\theta \,\mathrm{d}\,\theta}{\exp(\beta\,[mc^2\cosh\theta - \mu]) + 1}$

(d) **(10 pts)** $PV = \dfrac{8\pi V m^4 c^5}{3h^3} \displaystyle\int_0^{\infty} \frac{\sinh^4\theta \,\mathrm{d}\,\theta}{\exp(\beta\,[mc^2\cosh\theta - \mu]) + 1}$

(e) **(10 pts)** With the above results, and by taking the limit of $T \to 0$, show that for extremely relativistic particles (Fermi momentum $p_0 \to \infty$ and the corresponding θ_0 also approaches infinity), PV approaches $E/3$. (In this limit, E is dominated by kinetic energy K, therefore $PV = K/3$.)

You may need the integrals:

$$\int_{\theta=0}^{\theta_0} \sinh^2\theta\, \cosh^2\theta\, \mathrm{d}\,\theta = \frac{1}{32}\sinh(4\theta_0) - \frac{1}{8}\theta_0$$

$$\int_{\theta=0}^{\theta_0} \sinh^4\theta\, \mathrm{d}\,\theta = \frac{1}{32}\sinh(4\theta_0) - \frac{1}{4}\sinh(2\theta_0) + \frac{3}{8}\theta_0$$

■ **End of quesions.**