QM1 PhD Qualify

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1 Prob.1

Consider the 2 dimensional harmonic oscillator,

$$H = \sum_{\alpha=1}^{2} \left[\frac{p_\alpha^2}{2m} + \frac{m\omega^2 x_\alpha^2}{2} \right] = \hbar\omega \sum_{\alpha=1}^{2} [a_\alpha^\dagger a_\alpha + \frac{1}{2}] \tag{1}$$

- The operators $a_\alpha^\dagger, a_\alpha$ satisfy the commutation relations $[a_\alpha, a_\beta^\dagger] = \delta_{\alpha\beta}$. Please give the explicit expression of $a_\alpha^\dagger, a_\alpha$ in terms of x_α, p_β. (10 points)

- We define the vacuum as $a_\alpha|0\rangle = 0$. Explain why ? (5 pts)

- Use the fact that $[x_\alpha, p_\beta] = i\hbar\delta_{\alpha\beta}$ derive the vacuum wave function $\psi_0(x) = \langle x|0\rangle$. Please properly normalize (10 pts)

- Please derive the wave function for the the first excited states, how many are there (5 pts)? How many states are there for the second excited state (5 pts) ? The first and second excited states are defined as the states with lowest and second lowest energy state after the vacuum.

- Show that the operator $a_\alpha^\dagger a_\beta$ is a symmetry of the system. What is the symmetry ? (5 pts) (hint: SU(n))

- Define the state $|1_\alpha\rangle \equiv a_\alpha^\dagger|0\rangle$. Please compute $\langle 1_\alpha|p_\beta|0\rangle$ and $\sum_\beta \langle 1_\alpha|p_\beta^2|0\rangle$ (5 pts). One is vanishing and the other not. Please use symmetry arguments to explain why. (5 pts)

- The above symmetry is in fact the "same" as the three-dimensional rotation group, where the generators L_1, L_2, L_3 satisfies $[L_a, L_b] = i\epsilon_{abc}L_c$. Please express the generators L_a in terms of $(a_\alpha^\dagger, a_\beta)$. (5 pts)

2 Prob.2

Consider a system composed of two spin-$\frac{1}{2}$ particles that mutually interact with each other and also interact with an external uniform magnetic field $\mathbf{B} = -B\hat{\mathbf{k}}$. The Hamiltonian of the system is given by,

$$H = A\mathbf{S}_1 \cdot \mathbf{S}_2 + B(S_{1z} + S_{2z}). \tag{2}$$

- Given commutation relation $[S_i, S_j] = i\hbar \sum_{k=1}^{3} \epsilon_{ijk} S_k$, we can define the operators $S_\pm = S_x \pm iS_y$. Please show that

$$[S_z, S_\pm] = \pm\hbar S_\pm, \quad [S^2, S_\pm] = 0. \tag{3}$$

  (So that $S_\pm$ raise / lower the z component of the spin by $\hbar$ while preserving the total spin.) (10 pts)

- Find out what spin configurations of the system denote eigenstates of the Hamiltonian. (For example, is it $|+-\rangle$ or something else?) Choose one of the eigenstates and evaluate the energy expectation value. (10 pts)

3 Prob.3

Consider

$$a|\lambda\rangle = \lambda|\lambda\rangle$$

where a is the annihilation operator in the harmonic oscillator. This is a coherent state. The solution is

$$|\lambda\rangle = Exp(-|\lambda|^2/2)Exp(\lambda a^\dagger)|0\rangle \tag{4}$$

where $a^\dagger$ means the taking the adjoint of a, or equivalently, taking the transpose and then complex conjugate of a. Here $|0\rangle$ is the vacuum, i.e. $a|0\rangle = 0$

- Show that eq.(4) is indeed a solution. (10 pts)

- What is the expectation value for "photon number" $N = a^\dagger a$?. (5 pts)

- Please compute the square of the position and momentum deviation:

$$\begin{aligned} (\Delta X)^2 &= \langle X^2 \rangle - \langle X \rangle^2, \\ (\Delta P)^2 &= \langle P^2 \rangle - \langle P \rangle^2. \end{aligned} \tag{5}$$

  for the coherent state (10 pts) Use this to explain what is special about coherent states (5 pts)

4 Prob. 4

Consider the following state

$$\psi = \frac{1}{2}\left(|+\rangle + |-\rangle\right), \tag{6}$$

- Explicitly write out the following 2×2 density matrix (2 pts)

$$\rho_1 = |\psi\rangle\langle\psi|, \quad \rho_2 = \frac{1}{2}|+\rangle\langle+| + \frac{1}{2}|-\rangle\langle-|,$$

$$\rho_3 = \frac{1}{3}|+\rangle\langle+| + \frac{3}{4}(|-\rangle + |+\rangle)(\langle-| + \langle+|) \tag{7}$$

- One of these density matrix does not make any sense. Which one and why? (4 pts)

- Compute the von Neumann entropy $(-ktr\rho\ln\rho)$ for the remaining two. What is the maximal possible value for such two state system? (4 pts)