NTU-Physics Statistical Physics Qualifying Exam (2015)

(*Please note: 5 problems on 2 pages; Answers in both Chinese and English are OK.*)
The following formulas may be useful:

(1) $\sum_{\mathbf{k}} \rightarrow \frac{L^d}{(2\pi)^d}\int d^d k;\ \sum_{\mathbf{p}} \rightarrow \frac{L^d}{h^d}\int d^d p\ \ (V = L^d \rightarrow \infty)$, d is the dimensionality of thebox.

(2) $\int_0^\infty \frac{x^3}{e^x - 1}dx = \frac{\pi^2}{15},\quad \int_0^\infty x^2 e^{-\alpha x^2} dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}.$

(3) $\int_0^\infty \frac{\phi(x)dx}{e^{x-\xi}+1} = \int_0^\xi \phi(x)dx + \frac{\pi^2}{6}\left(\frac{d\phi}{dx}\right)_{x=\xi} + \frac{7\pi^4}{360}\left(\frac{d^3\phi}{dx^3}\right)_{x=\xi} + \cdots$

1. (20 %) (a) (7 %) Assuming that the total number of microstates accessible to a given statistical system is Ω, show that the entropy of the system, as given by $S = -k\langle \ln P_r \rangle = -k\sum_r P_r \ln P_r$, is maximum when all Ω states are equally likely to occur.

(b) (7 %) If, on the other hand, we have an ensemble of systems sharing energy (with mean value $\bar{E}$), show that the entropy, as given by the same expression, is maximum when $P_r \propto \exp(-\beta E_r)$, β being a constant to be determined by the given value of $\bar{E}$.
(c) (6 %) Further, if we have an ensemble of systems sharing both energy and particles (with mean value $\bar{N}$), show that the entropy, given by a similar expression, is maximum when $P_{r,s} \propto \exp(-\alpha N_r - \beta E_s)$, α and β being constants to be determined by the given values of $\bar{N}$ and $\bar{E}$.

2. (20 %) An ideal classical gas composed of N particles, each of mass m, is enclosed in a vertical cylinder of height L placed in a uniform gravitational field (of acceleration g) and is in thermal equilibrium; ultimately, both N and $L \rightarrow \infty$.
(a) (12 %) Evaluate the partition function of the gas and derive expressions for its major thermodynamic properties (at least, U, C_V).
(b) (8 %) Explain why the specific heat of this system is larger than that of a corresponding system in free space.

3. (20 %) The internal energy of a photon gas in a cavity of V and at temperature T is given by $U(T,V) = \sum_{\mathbf{k},\hat{e}} \hbar\omega \langle n_{\mathbf{k},\hat{e}} \rangle$ where $\langle n_{\mathbf{k},\hat{e}} \rangle$ is the average number of photons in quantum state $\mathbf{k}$, frequency $\omega_{\mathbf{k}} = c|\mathbf{k}|$ and electric field polarization $\hat{e}$.

(a) (7 %) Calculate the internal energy U and the specific heat C_V.

(b) (7 %) Calculate the entropy S and the average number of photons $<N>$ in the cavity. What is the entropy per photon s (= $S/<N>$)?

(c) (6 %) When $T\to\infty$, what is the specific heat per unit volume c_V (= C_V/V)? Why?

4. (20 %) (a) (10 %) Show that, quite generally, the low-temperature behavior of the chemical potential, the specific heat, and the entropy of an ideal Fermi gas are given, respectively, by $\mu \simeq \varepsilon_F \left[1 - \frac{\pi^2}{6}\left(\frac{\partial \ln g(\varepsilon)}{\partial \varepsilon}\right)_{\varepsilon=\varepsilon_F}\left(\frac{kT}{\varepsilon_F}\right)^2\right]$, and $C_V \simeq S \simeq \frac{\pi^2}{3}k^2 T g(\varepsilon_F)$, where $g(\varepsilon)$ is the density of (the single-particle) states in the system.

(b) (10 %) Examine these results for a gas with energy spectrum $\varepsilon \propto p^s$, confined to a space of n dimensions and discuss the special cases: s = 1 and 2, with n = 2 and 3.

5. (20 %) General questions.

[Simple answers (for example, a number, a yes or a no or a phrase) will be sufficient.]

(a) (5 %) Consider an ideal gas consisting of monatomic diatomic molecules such as O_2 and H_2. The molecules can have a bond-stretching vibration, and therefore, they have both the translational and vibrational degrees of freedom. However, at normal conditions, we usually find that the ambient pressure specific heat of the gas is $C_P = (5/2)Nk$ instead of $C_P = (7/2)Nk$. Why?

(b) (5 %) Both photons and phonons are bosons. However, there is no Bose-Einstein condensation in a photon gas and a phonon gas. Why?

(c) (5 %) A white dwarf star can be considered as a gas composed of helium nuclei and electrons which are fermions. Unlike a boson gas, a fermion gas must have a considerable kinetic energy even at very low temperature and hence could exert an enormous pressure which actually prevents many white dwarf stars from gravitationally collapsing into a black hole due to the mass of helium nuclei. What is the quantum mechanical principle which stops the electron gas having a zero kinetic energy at zero temperature?

(d) (5 %) A metal (e.g., copper) consists of light free electrons and heavy ions. However, when we measure the specific heat of the metal at room temperature, we generally find that the specific heat $C_V = 3N_{ion}k$ which solely comes from the ionic vibrations according to the classical equipartition theorem. Why do we not find the contribution of $(3/2)N_e k$ from the free electron gas in the metal according to the equipartition theorem too? Here N_{ion} and N_e denote the numbers of the ions and free electrons in the metal, respectively.