NTU-Physics Statistical Physics Qualifying Exam (2018)

(*Please note: 5 problems on 2 pages; Answers in both Chinese and English are OK.*)
The following formulas may be useful:

(1) $\sum_{\mathbf{k}} \rightarrow \frac{L^d}{(2\pi)^d}\int d^d k;\ \sum_{\mathbf{p}} \rightarrow \frac{L^d}{h^d}\int d^d p\ \ (V = L^d \rightarrow \infty),\ d$ is the dimensionality of the box.

(2) $I_v \equiv \int_0^\infty e^{-\alpha y^2} y^v dy = \begin{cases} (1/2)\sqrt{\pi/\alpha} & \text{for } v = 0 \\ (1/4)\sqrt{\pi/\alpha^3} & \text{for } v = 2 \end{cases}$

(3) Bose-Einstein integrals $g_n(z) = \frac{1}{\Gamma(n)}\int_0^\infty \frac{x^{n-1}dx}{z^{-1}e^x - 1} = \sum_{k=1}^\infty \frac{z^k}{k^n}\quad (0 \le z \le 1),\quad \Gamma(n+1) = n\Gamma(n)$

If $z = 1\ (\mu = 0),\ g_n(1) = \zeta(n)\quad n > 1;$ if $z \rightarrow 1,\ g_n(z) \rightarrow \infty\quad n \le 1.$

(4) $\cosh(x) = (\frac{e^x + e^{-x}}{2}),\ \sinh(x) = (\frac{e^x - e^{-x}}{2});\ \cosh(x)' = \sinh(x),\ \sinh(x)' = \cosh(x).$

1. (20 %) An ideal classical gas composed of N particles, each of mass m, is enclosed in a vertical cylinder of height L placed in a uniform gravitational field (of acceleration g) and is in thermal equilibrium; ultimately, both N and $L \rightarrow \infty$.
(a) (12 %) Evaluate the partition function of the gas and derive expressions for its major thermodynamic properties (at least, U, C_V).
(b) (8 %) Explain why the specific heat of this system is larger than that of a corresponding system in free space.

2. (20 %)
(a) (10 %) Evaluate the partition function and the major thermodynamic properties (A, S, U) of an ideal gas consisting of N_1 molecules of mass m_1 and N_2 molecules of mass m_2, confined to a space of volume V at temperature T. Assume that the molecules of a given kind are mutually indistinguishable, while those of one kind are distinguishable from those of the other kind.
(b) (10 %) Compare your results with the ones pertaining to an ideal gas consisting of ($N_1 + N_2$) molecules, *all of one kind*, of mass m, such that $m(N_1+N_2) = m_1N_1 + m_2N_2$.

3. (20 %) In the harmonic approximation, the lattice vibration energy of a solid can be approximated as the internal energy of a phonon gas in the solid of V and at temperature T given by $U(T,V) = \sum_i \hbar\omega_i(1/2 + \langle \hat{n}_i \rangle)$ where $\langle \hat{n}_i \rangle$ is the average number of phonons in quantum state i. Einstein assumed, for simplicity, that all the frequencies are equal, i.e.,

$\omega_i = \omega_E$.

(a) (7 %) Derive the general expression for the heat capacity C_V.

(b) (6 %) Derive the expression for C_V in the Einstein approximation.

(c) (7 %) Calculate C_V in (b) in the high T [$T \gg \theta_E = (\hbar\omega_E / k)$] and low T ($T \ll \theta_E$) limits.

4. (20 %) Consider an ideal Bose gas in an isotropic *two-dimensional* harmonic trap.

(a) (10 %) Determine the density of states $g(\varepsilon)$, and express the number of particles in the excited states, N_e, and the number of particles in the ground state, N_0.

(b) (10 %) Can a Bose-Einstein condensate form in this trap? If so, what is the Bose-Einstein condensation temperature T_c and the condensate fraction N_0/N for $T \le T_c$? If not, why?

5. (20 %) The partition function of the one-dimensional Ising model is given by

$Q_I(B,T) = \sum_{\sigma_1}\sum_{\sigma_2}\cdots\sum_{\sigma_N}\exp[\beta\sum_{k=1}^{N}(J\sigma_k\sigma_{k+1} + B\sigma_k)]$. Introduce a 2×2 transfer matrix P such

that $\langle\sigma|P|\sigma'\rangle = \exp\{\beta[J\sigma\sigma' + B(\sigma+\sigma')/2]\}$, where ($\sigma$, σ' = ±1). One can solve the 1-D

Ising model with this transfer matrix by assuming the periodic boundary condition $\sigma_{N+1} = \sigma_1$.

(a) (8 %) Calculate the partition function $Q_I(B,T)$.

(b) (6 %) Calculate the Helmholtz free energy $A_I(B,T)$.

(c) (6 %) Calculate the magnetization $M_I(B,T)$ and show that for any nonzero temperature, there is no spontaneous magnetization in the one-dimensional Ising model.