NTU Examination of PhD Qualifacation (March 3-4, 2023)
Classical Mechanics

In the following, $\hat{e}_x,\ \hat{e}_y,\ \hat{e}_z$ are the unit vectors along the $x,\ y,\ z-$axes, respectively.

1. A disk of mass m and radius r_0 is originally vertically on the top of an inclined plane of total length l_0 fixed with an angle ϕ to the horizontal ground. Let us call the original contact point between the disk and the plane point P, the center of the disk point O. At time $t = 0$, the disk starts to roll down without slipping. At a very short time period dt, the disk travels a distance dx on the plane. The new contact point between both becomes point P', and the angle between lines OP and OP' on the disk is $d\theta$. (The moment of inertia for a disk is $I = \frac{1}{2}mr^2,\ r$ being its radius).
   (a) Introducing the Lagrange undetermined multiplier λ to treat the constraint, set up the Lagrange's equations. Solve them. (8%)
   (b) Find the speed of the disk when it arrives the plane's bottom. What is its total mechanical energy at this moment? How much energy does it lose during the travel? (6%)

2. The coordinate and momentum of a system of mass m are transformed from $(q,\ p)$ to $(Q,\ P)$ via $Q = p + iaq$ and $P = \frac{p-iaq}{2ia}$, with $a = \sqrt{mk}$, k being a constant.
   (a) Apply the Poisson brackets to show the transformation being canonical. It is apparently $AQP = H$, H the system's Hamiltonian. What is A? (7%)
   (b) In the new coordinate-momentum system, solve $Q,\ P$, and thus find $q,\ p$ in functions of time. (7%)

3. A particle of mass m and energy E moving in one dimensional space under the potential $V = kx$ with $k > 0$ as $x \geq 0$ and $V \to \infty$ as $x < 0$.
   (a) Find the Hamilton's principal function $S(x, t)$ and from it write down the wave function for $x \geq 0$. (6%)
   (b) Find the action variable and then find the particle's frequency in the motion. (8%)

4. A point object of mass m is fastened to a massless string of length $3l$ with the string's another end suspended to the ceiling. An identical object of the same mass is hung from the first object through the massless string of length $4l$. The system is originally mottinless in its equilibrium position lying on the $z-$axis perpendicular to the ground.
   (a) The system can be set to perform small oscillations about equilibrium. Let us set θ_1 the angle between the top string to the $z-$axis and θ_2 that between the lower string and the $z-$axis during oscillations. Apply the fact that $sin\theta \approx \theta$ and keeping only the first order terms of $\theta_i,\ i = 1, 2$, set the equations of motion for the system. (6%)
   (b) Solve equations obtained from part (a), find the normal mode frequencies and normal mode coordinates. (9%)

To be continued in next page

5. (a) A rigid body contains N particles, with the i^{th} particle of mass m_i and position vector $\vec{r_i} = x_i\hat{e}_x + y_i\hat{e}_y + z_i\hat{e}_z$ $(i = 1, ..., N)$. Its angular momentum is $\vec{L}$ when rotates about a certain axis at the angular velocity $\vec{\omega} = \omega_x\hat{e}_x + \omega_y\hat{e}_y + \omega_z\hat{e}_z$. Write down each component of the inertia tensor in terms of $m_i,\ x_i,\ y_i,\ z_i$. (6%)

   (b) A certain rigid body may be replaced by the three identical point particles each of mass m_0, at $(0, 1, 1)$, $m_2 = 2m_0$ at $(1, 1, 0)$, and $(1, 0, 1)$, here coordinates in the unit of r_0= constant, that is, "1" means "r_0", "2" means "$2r_0$", and so on. Find the principal moments of inertia about the origin and a set of principal axes. (9%)

6. In a laboratory, a particle of rest mass m_1 with total energy E_1^L (here $E_1^L - m_1c^2 > 0.5m_1c^2$) hits on a motionless particle of rest mass m_2. You may set the incident direction in the $x-$axis. After the collision, they become particles of rest masses m_3 and m_4. Let $p_i^L,\ p_i^C$ be the i^{th} particle's 4-momenta in the lab and the center of mass (CM) system. Define $s = (p_1+p_2)^2 = (p_3+p_4)^2;\ t = (p_1-p_3)^2 = (p_2-p_4)^2;\ u = (p_1-p_4)^2 = (p_2-p_3)^2$, all p_i's being 4-momenta. These definitions are true for both the lab and the CM frames since they are all invariants, $P^2 = P^\mu P_\mu$.

   (a) Write down $s,\ t,\ u$ in both the laboratory and the CM system. (7%)

   (b) Show that $s + t + u = (m_1^2 + m_2^2 + m_3^2 + m_4^2)c^4$. (7%)

7. (a) Prove that if the only force exerted on a system is a central force, the system's angular momentum $\vec{L}$ is a constant. Hence prove that its position vector from the force center $\vec{r}$ is always on the plane perpendicular to $\vec{L}$. (7%)

   (b) Because of part (a), you can write the Lagrange equation for the system in two-dimensional polar coordinates r, θ. Find the central force that results to the orbit of a particle of mass m as $r\theta = a,\ a$ being a constant. Your answer should be a function of r only, with some constants. (7%)