Statistical Physics Qualifying Exam (2023)

Please note: 4 problems on 2 pages. Answers in Chinese or English are acceptable.

1. **30pts.** Consider N non-interacting indistinguishable quantum particles whose wavefunction can be expressed by $\Psi^{(N)}_{\vec{k}_1,\vec{k}_2,\cdots}(\vec{r}_1,\vec{r}_2,\cdots) = (N!)^{-1/2}\sum_P \delta_P u_{\vec{k}_1}(\vec{r}_1)u_{\vec{k}_2}(\vec{r}_2)\cdots$ with $u_{\vec{k}}(\vec{r}) = V^{-1/2}e^{i\vec{k}\cdot\vec{r}}$ describing the single-particle wavefunction at the coordinate $\vec{r}$ and of wavevector $\vec{k}$, where V is the 3-dimensional quantization volume. The numeric subscripts go from 1 to N corresponding to the particle index and P is the particle permutation operator with $\delta_P = +1$ for bosons, and $\delta_P = +1$ if P is even permutation and $\delta_P = -1$ if P is odd permutation for fermions. For free particles of mass m, the Hamiltonian is given by $H = \frac{\hbar^2}{2m}\sum_{i=1}^{N} k_i^2$.

(a) (**10pts**) For $N = 2$, show that

$$\langle \vec{r}_1,\vec{r}_2|e^{-\beta H}|\vec{r}_1,\vec{r}_2\rangle \equiv \frac{V^2}{(2\pi)^6}\int d^3k_1 d^3k_2 \left\langle \Psi^{(2)}_{\vec{k}_1,\vec{k}_2}(\vec{r}_1,\vec{r}_2)\right| e^{-\beta H}\left|\Psi^{(2)}_{\vec{k}_1,\vec{k}_2}(\vec{r}_1,\vec{r}_2)\right\rangle$$
$$= \frac{1}{2\lambda^6}\left[1 \pm \exp\left(-\frac{2\pi r_{12}^2}{\lambda^2}\right)\right],$$

where "$+$" is for bosons and "$-$" for fermions with $\beta = 1/(k_B T)$ for temperature T, the de Broglie wavelength $\lambda = \hbar\sqrt{2\pi\beta/m}$, and $r_{12} = |\vec{r}_1 - \vec{r}_2|$. You may need the Gaussian integral $\int_{-\infty}^{\infty}\exp(-\alpha x^2)\,dx = \sqrt{\pi/\alpha}$.

(b) (**10pts**) Following the above, show that the partition function is given by

$$Z(V,T) = \frac{1}{2}\left(\frac{V}{\lambda^3}\right)^2\left[1 \pm \frac{1}{2^{3/2}}\left(\frac{\lambda^3}{V}\right)\right],$$

and that the correlation function

$$\langle \vec{r}_1,\vec{r}_2|\rho|\vec{r}_1,\vec{r}_2\rangle \approx V^{-2}[1 \pm \exp(-2\pi r_{12}^2/\lambda^2)]$$

in the thermodynamic limit.

(c) (**10pts**) For 2 non-interacting classical particles, show that

$$\langle \vec{r}_1,\vec{r}_2|\rho|\vec{r}_1,\vec{r}_2\rangle = V^{-2}.$$

Comparing the results of (b) and (c), what does the correction term $\pm\exp(-2\pi r_{12}^2/\lambda^2)$ imply? [Hint: You may write $\langle\rho(r)\rangle \propto \exp[-\beta u(r)]$ and discuss the meaning of $u(r)$.]

2. **20pts.** Starting with the definition of the internal energy $U \equiv \sum_i p_i E_i$, where p_i is the probability of the system in the ith state of energy E_i, "prove" the first law of thermodynamics from the microscopic point of view by identifying the heat and work along with their corresponding microscopic interpretation.

3. **25pts.** Consider an ideal classical gas contained in a box. Its walls have n_0 absorbing sites, each of which can capture at most one molecule of the gas. Let $-\epsilon$ be the energy of an absorbed molecule. Suppose the gas and the walls are in thermal (and particle) equilibrium at temperature T and the pressure of the gas is P.

(a) (**10pts**) Find the fugacity $z = e^{\beta\mu}$ of the gas, where μ is the chemical potential.

(b) (**15pts**) Find the mean number of the absorbed molecules and investigate its low- and high-pressure limits.

4. **25pts.** The energy levels of a quantum rigid rotor are $\epsilon_j = \frac{j(j+1)h^2}{8\pi^2 ma^2}$ with $j = 0,1,2,\cdots$ with a degeneracy of each level $g_j = 2j+1$, where ma^2 corresponds to the moment of inertia and h is the Plank constant. Define $\theta \equiv \frac{h^2}{8\pi^2 ma^2 k_B}$.

(a) (**5pts**) Find the general expression for the partition function.

(b) (**10pts**) In the high-temperature limit, where $\theta/T \ll 1$, show that the partition function can be approximated by

$$Z \approx \int_0^\infty e^{-\theta x/T} dx \approx \frac{T}{\theta} = \frac{8\pi^2 ma^2 k_B T}{h^2}.$$

Find the internal energy and heat capacity.

(c) (**10pts**) In the low-temperature limit, only keep the ground state and the first excited state to approximate the partition function. Find the internal energy and heat capacity.