QM1 PhD Qualify

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1 Prob.1

Consider the 4 dimensional harmonic oscillator,

$$H = \sum_{\alpha=1}^{4} \left[\frac{p_\alpha^2}{2m} + \frac{m\omega^2 x_\alpha^2}{2} \right] = \hbar\omega \sum_{\alpha=1}^{4} [a_\alpha^\dagger a_\alpha + \frac{1}{2}] \tag{1}$$

The energy eigenstates have energy eigenvalue

$$E_N = \hbar\omega[N+2] \tag{2}$$

where $N = n_1+n_2+n_3+n_4$, and n_i is the excitation number in each direction.

- The operators $a_\alpha^\dagger, a_\alpha$ satisfy the commutation relations $[a_\alpha^\dagger, a_\beta] = \delta_{\alpha\beta}$. Please give the explicit expression of $a_\alpha^\dagger, a_\alpha$ in terms of x_α, p_β. (10 points)
- What is the degeneracy number for states with E_1, E_2, E_3 energy eigenvalues (2 pts) and general N (5 pts) ?
- Please explain the occurrence of the degeneracy in terms of the symmetry of the Hamiltonian (5 pts) ?
- Construct the operator that corresponds to the generator of this symmetry. Said in another way, construct the operators that commute with the Hamiltonian yet induces linear transformations amongst the degenerate eigenstates (13 pts).

2 Prob.2

For a particle in an electromagnetic background, the action takes the form:

$$S = \int d\tau \left(\frac{m}{2} \dot{x}_i \dot{x}_i + e\dot{x}_i A_i \right), \qquad i = 1,\ 2,\ 3$$

We will consider a monopole back ground where the gauge potential is given as

$$\vec{A} = (A_r, A_\theta, A_\phi) = (0, 0, \frac{e_m(1-\cos\theta)}{r\sin\theta}) \tag{3}$$

- From the action please derive the canonical momentum (5 pts)
- (a) Using that in polar coordinates

$$\begin{aligned}\nabla \times \vec{V} &= \hat{r}\left[\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(V_\phi \sin\theta) - \frac{\partial}{\partial\phi}V_\theta\right] + \hat{\theta}\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}V_r - \frac{\partial}{\partial r}(rV_\phi)\right] \\ &+ \hat{\phi}\frac{1}{r}\left[\frac{\partial}{\partial r}(rV_\theta) - \frac{\partial}{\partial\theta}V_r\right] \qquad (4)\end{aligned}$$

  Show that the gauge potential in eq.(3) correspond to a magnetic field sourced by a magnetic charge proportional to e_M. [5 pts]
- The gauge potential has a singularity at $\theta = \pi$. Is this a physical singularity ? Please explain [5 pts]
- Show that, if we naively treat the angular momentum operator L_i in this system as $L_i = \sum_{j,k}\epsilon_{ijk}x_j\dot{x}_k$, they will *not* satisfy the SO(3) algebra ($[L_i, L_j] = -\epsilon_{ijk}L_k$) [10 pts].
- Find out the suitable modification of the above angular momentum operator such that $[L_i, L_j] = -\epsilon_{ijk}L_k$ is satisfied [10 pts].

3 Prob.3

The coherent states are defined as the eigenstates of lowering operator, a, as

$$a|\alpha\rangle = \alpha|\alpha\rangle.$$

1. Please compute the square of the position and momentum deviation:

$$\begin{aligned}(\Delta X)^2 &= \langle X^2\rangle - \langle X\rangle^2, \\ (\Delta P)^2 &= \langle P^2\rangle - \langle P\rangle^2. \qquad (5)\end{aligned}$$

   for the coherent state [10 pts]

2. Derive the following relation

$$|\alpha\rangle = \exp^{\{\hat{\alpha}a^\dagger - \hat{\alpha}^* a\}}|0\rangle \qquad (6)$$

   where $\hat{\alpha}$ and $\hat{\alpha}^*$ are complex numbers and $a, a^\dagger$ are the annihilation and creation operators of a one dimensional simple harmonic oscillator and $|0\rangle$ the ground state. [10 pts]

3. Derive the wavefunction of the $|\alpha\rangle$-state by projecting Eq. (6) on position eigenstates so that you have a differential equation . [10 pts]