Spring 2019

Qualifying Exam — Quantum Mechanics I

Instructions: This is a closed-book test. There are three sets of problems with scores indicated in front of each subproblem. No calculator usage is allowed during the exam.

1. (20%) Consider a free particle in one dimension with mass m. At time $t = 0$ the expectation value of its position is $\langle x \rangle_0$ with a variance $(\Delta x)_0^2 = \langle x^2 \rangle_0 - \langle x \rangle_0^2$, where the subscript 0 refers to $t = 0$. Find the variance at some later time t. Express your answer in terms of t and expectation values of operators at $t = 0$, including the operators x, momentum p, and combinations thereof.

2. Consider a one-dimensional simple harmonic oscillator of mass m and charge q. Suppose the system is placed in a static electric field of strength E. The Hamiltonian of this oscillator is given by

$$\hat{H} = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 - qEx \, .$$

The ground-state wave function when $E = 0$ is given by

$$\psi_0(x) = \frac{1}{(\sqrt{\pi} x_0)^{1/2}} \exp\left[-\frac{1}{2} \left(\frac{x}{x_0} \right)^2 \right] \, , \quad \text{with } x_0 \equiv \sqrt{\frac{\hbar}{m\omega}} \, .$$

   (a) (10%) For a constant electric field E, find the energy levels for all states.

   (b) (15%) Determine the most likely position of the oscillator in the ground state and give a physical interpretation for your result.

   (c) (15%) Now consider the case where the electric field is an oscillating field, $E = E_0 \cos(\omega' t)$, with ω' different from ω in general. Go to the Heisenberg picture. Compute the following quantities

$$\frac{dx}{dt} \text{ and } \frac{dp}{dt} \, .$$

   (d) (10%) Solve for the position operator $x(t)$ with the initial conditions:

$$x(0) = x_0 \text{ and } \left. \frac{dx}{dt} \right|_{t=0} = 0$$

3. Consider a beam of atoms with some specific total angular momentum j in the famous Stern-Gerlach experiment. Suppose we use the notation SG$\hat{\mathbf{n}}$ to denote a Stern-Gerlach apparatus with an inhomogenous magnetic field along the $\hat{\mathbf{n}}$ direction.

(a) (10%) Suppose the atoms in the beam with total angular momentum j all have initially a specific value for the component of angular momentum along the $\hat{\mathbf{z}}$ axis. How many component beams does one get after the beam passes through an SG$\hat{\mathbf{n}}$ (assuming $\hat{\mathbf{n}} = (\theta \neq 0, \phi = 0)$, where θ is the polar angle and ϕ is the azimuthal angle)? Explain qualitatively the relative intensities of the component beams.

(b) (10%) Suppose now $j = 1/2$ and the atoms in the beam are initially polarized in the $+\hat{\mathbf{z}}$ direction (*i.e.*, all spin-up), determine the relative intensities of the component beams.

(c) (10%) Show that if the angular momentum j system is in an eigenstate of J_z, then the expectation values of the other two components of total angular momentum operator J_x and J_y vanish.