2014 Ph. D. Qualifying Exam ---- Quantum Mechanics I

1. Let Ω and Λ be two Hermitian operators, with a commutator $[\Omega, \Lambda] = i\Gamma$.
   (a) Please show $(\Delta\Omega)^2(\Delta\Lambda)^2 \geq \frac{1}{4}\langle\psi|\,[\Omega, \Lambda]_+|\psi\rangle^2 + \frac{1}{4}\langle\psi|\Gamma|\psi\rangle^2$. (10 pts) (hint: use the Schwarz inequality)
   (b) i) Show that if $[\Omega, \Lambda] = i\hbar$, one has $(\Delta\Omega)\cdot(\Delta\Lambda) \geq \hbar/2$, ii) Compute $(\Delta T)\cdot(\Delta X)$, where $T = P^2/2m$. (10 pts)
2. (a) Consider a particle of mass m moving freely between $x = 0$ and $x = a$ inside an infinite square well potential. If the particle was in the ground state. Suddenly the right-hand side is moved to $x = 2a$. What is the probability that the particle will be in the ground state of the new potential? (10 pts) (b) Consider a one-dimensional particle which is confined within the region $0 \leq x \leq a$ and whose wave function is $\Psi(x, t) = \sin(\pi x/a)\exp(-i\omega t)$. Find the potential $V(x, t)$ (10 pts)
3. Consider a system of three non-interacting identical spin ½ particles that are at the same spin state $|\frac{1}{2}, \frac{1}{2}\rangle$ and confined to move in a one-dimensional infinite potential well of length a: $V(x) = 0$ for $0 < x < a$ and $V(x) = \infty$ for other values of x. Determine the energy and wave function of the ground state, the first excited state, and the second excited state. (20 pts)
4. (a) The electron in the hydrogen atom is in a state described by the wave function $\frac{1}{6}[3\psi_{100}(\boldsymbol{r}) + 4\psi_{211}(\boldsymbol{r}) - \psi_{210}(\boldsymbol{r}) + \sqrt{10}\,\psi_{21-1}(\boldsymbol{r})]$, calculate the expectation values of the energy, L^2, and L_z. (10 pts) (b) Consider a state $|\psi\rangle = \frac{1}{\sqrt{2}}|\varphi_1\rangle + \frac{1}{\sqrt{5}}|\varphi_2\rangle + \frac{1}{\sqrt{10}}|\varphi_3\rangle$ which is given in terms of three orthonormal eigenstates $|\varphi_1\rangle$, $|\varphi_2\rangle$ and $|\varphi_3\rangle$ of an operator $\hat{O}$ such that $\hat{O}|\varphi_n\rangle = n^2|\varphi_n\rangle$. Find the expectation value of $\hat{O}$ for the state $|\psi\rangle$. (10 pts)
5. (a) Let $|n\rangle$ be an eigenstate of the 1-dim harmonic oscillator (1DHO) potential. Please calculate the expectation value of the operators $\boldsymbol{P}^2$ and $\mathbf{X}^3$ in the N-representation with the state $|n\rangle$ (i.e., $\langle n|\boldsymbol{P}^2|n\rangle$ and $\langle n|\boldsymbol{X}^3|n\rangle$). (10 pts)
   (b) The electron's spin wave function is an eigenstate of $\mathbf{S}_z$ with eigenvalue $+\hbar/2$. The operator $\hat{\mathbf{e}}\cdot\mathbf{S}$ represents the spin projection along a direction $\hat{\mathbf{e}}$. We can express this direction as $\hat{\mathbf{e}} = \sin\theta(\cos\varphi\,\mathbf{i} + \sin\varphi\,\mathbf{j}) + \cos\theta\,\mathbf{k}$.($\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are unit vectors of the x-, y-, and z-axes) What is the probability of finding the electron in each $\hat{\mathbf{e}}\cdot\mathbf{S}$ eigenstate**?** (10 pts)

   [hint: $\boldsymbol{a} = \sqrt{\frac{mw}{2\hbar}}\boldsymbol{X} + i\sqrt{\frac{1}{2mw\hbar}}\boldsymbol{P}$; $\boldsymbol{a}^+ = \sqrt{\frac{mw}{2\hbar}}\boldsymbol{X} - i\sqrt{\frac{1}{2mw\hbar}}\boldsymbol{P}$)]